Journal of Applied Mathematics and Computing (2025) 71:4787–4813
https://doi.org/10.1007/s12190-025-02409-z

ORIGINAL RESEARCH

Determining the type of a solution to the fully pythagorean fuzzy linear equations system: exact, restricted, or relaxed approximate solution

Gizem Temelcan Ergenecosar[1]

Received: 5 September 2024 / Revised: 30 January 2025 / Accepted: 31 January 2025 /
Published online: 15 February 2025

Abstract

In engineering and social research, linear systems are commonly used to address real-life problems of various dimensions. Therefore, many studies start by developing linear systems and then finding their solutions. Recent studies have demonstrated the effectiveness of Pythagorean fuzzy sets in capturing and representing complex forms of uncertainty, particularly when understanding the distinctions between membership and non-membership is crucial. This paper pioneers the finding a solution for a general (square or nonsquare) Fully Pythagorean Fuzzy Linear Equations System (FPFLES) with arbitrary triangular Pythagorean fuzzy numbers and fills a critical gap in the existing literature. Since an FPFLES consists of the sum of the multiplications of each arbitrary parameter and variable, and the fuzzy multiplication operation includes the min and max operators, a nonlinearity situation is observed in each equation. To overcome this situation, a transformation from fuzzy multiplication to inequalities is applied, and thus, a mixed integer programming (MIP) problem is formed. Depending on whether the MIP problems created by changing the constraints have an optimal solution, FPFLES has an exact solution or an approximate solution. The types of solutions are examined using a distance measure definition available in the literature. This paper also defines restricted and relaxed approximate solutions for FPFLES by determining whether the left-hand sides obtained from the substitution of solutions are completely covered by the right-hand sides of the equations. The approach is illustrated with some numerical examples, and the numerical results are analyzed within the distance measure to determine the closeness between the left-hand and right-hand sides of the system.

Keywords Fully fuzzy linear equations system · Triangular Pythagorean fuzzy numbers · Mixed integer programming · Fuzzy Pythagorean solution

✉ Gizem Temelcan Ergenecosar
temelcan.gizem@gmail.com

1 Department of Computer Engineering, Beykoz University, Kavacik, Beykoz, 34010 Istanbul, Turkey

1 Introduction

Given the widespread usage of linear assumptions in science and engineering, many studies begin with the creation of linear models. The majority of advanced issues also necessitate the use of linear systems, often with large dimensions. Basic sciences such as mathematics and physics; social science; and engineering require LES. The formula for a typical LES is $AX = b$, where A is a matrix of parameters, b is a constant vector and X is a variable vector.

In many applications, however, the parameters of an LES cannot be precisely defined. This is primarily due to a lack of information, changing conditions, and real-life problems involving uncertain situations. Moreover, errors made during the data collection process can result in the loss of valuable information. Therefore, using fuzzy numbers in LES helps to model real-life problems more realistically. To deal with uncertain information, Zadeh [52] first introduced the concept of fuzzy sets, after which the use of fuzzy sets and fuzzy numbers increased rapidly in several areas such as medicine [16, 47], engineering [38, 43], and social sciences [11, 40].

Allahviranloo and Mikaeilvand [9] discussed methods for finding positive fuzzy number solutions to FFLES to handle uncertainties in real-world problems. Then, they [10] presented methods for finding non-zero fuzzy number solutions to FFLES. Nayak and Chakraverty [37] proposed a new arithmetic approach and algorithms to solve FFLES. Liu [32] presented a new approximate method for solving FFLES and observed that the numerical results they found were more accurate than the iterative Jacobi and Gauss-Seidel methods. Kumar and Bansal [29] proposed a method to solve FFLES with arbitrary coefficients and an arbitrary solution vector, i.e. there is no restriction on the elements of the system. The approach integrates the concepts of linear programming to solve FFLES with arbitrary coefficients. Ezzati et al. [20] proposed a method to solve square FFLES by converting them to crisp linear systems and claimed that their proposed method always finds a solution to the system. Moloudzadeh et al. [35] proposed a method working only for triangular fuzzy numbers to solve an arbitrary square FFLES using 0-cut and 1-cut, and established the necessary and sufficient conditions for the uniqueness of a fuzzy solution. Ahlatcioglu et al. [4] and Kocken et al. [27] constructed a MIP problem to find the feasible fuzzy solution of a square FFLES with triangular fuzzy numbers. Their novelty was removing all the sign restrictions on the parameters and variables. Mikaeilvand et al. [34] applied a technique based on the embedding method to solve the square fuzzy LES, and obtained the strong fuzzy number solutions.

Atanassov [12] extended the concept of fuzzy sets and defined intuitionistic fuzzy sets. An application of intuitionistic fuzzy sets can be presented as intuitionistic fuzzy LES, and [13, 14, 39, 44, 46] are some studies including methods developed to solve these systems. To manage the complex impreciseness and uncertainty, Yager [50] defined the PFS as an extension of intuitionistic fuzzy sets. The main characteristic of the Pythagorean fuzzy concept is the condition that the square sum of its membership degree and non-membership degree is not greater than 1. In recent studies, it is claimed that the PFS has certain advantages in specific contexts. One of these aspects is that the PFSs are more flexible in defining membership degrees, i.e. having membership and non-membership functions. Another advantage is the successful application of the PFSs, particularly in decision-making problems where a detailed representation of information produces superior results. Also, PFSs allow for the representation of the hesitancy degree (which reflects uncertainty) as a square root formulation, which offers a finer granularity than the hesitance degree in intuitionistic fuzzy numbers. Given these aspects, using a PFS is more convenient than using intuitionistic fuzzy sets. After the definition of the PFSs, Zhang and Xu [53] presented a PFN, and then the basic operational rules of PFNs and Pythagorean fuzzy aggregation operators. Luqman et al. [33] defined triangular PFN and used them in the digraph and matrix approach. After the definitions and propositions, PFNs began to be used in decision-making [21, 22, 42, 48, 49] and optimization problems [5–7, 24, 30]. To our knowledge, no prior research has tackled solving an FPFLES, making this study the first of its kind.

In this paper, an FPFLES is considered, and the types of the Pythagorean fuzzy solution are studied. The main contribution of this paper is as follows:

- The concept of an FFLES is extended in the Pythagorean fuzzy environment based on triangular PFNs.
- The FPFLES is constructed with arbitrary (non-negative, mixed, or negative) triangular PFNs and variables in triangular Pythagorean fuzzy form.
- The FPFLES can be in a square or non-square structure.
- Identifying the minimum and maximum elements of a finite set through an MIP problem is implemented as the basic idea.
- To overcome the non-linearity situation caused by the multiplication of two arbitrary triangular PFNs, a transformation from the fuzzy multiplication to the set of crisp inequalities is used considering the study [27].
- Using the concept of fuzzy equality, the paper demonstrates that the FPFLES and the MIP problem, including the inequalities obtained from fuzzy multiplication, are equivalent, and thus they have identical solutions.
- The initial aim is to yield an exact solution for an FPFLES. If the system fails to find an exact solution, it focuses on finding a restricted or relaxed approximate solution by adjusting the MIP problem's constraints.
- A distance measure is stated for the Pythagorean fuzzy numbers, and considering the distance function proposed by Ebadi et al. [19], the closeness between the left-hand and right-hand sides of the FPFLES is calculated.
- The adapted approach is applied to solve some practical problems.

The paper is arranged as follows: Sect. 2 presents some necessary preliminaries. Section 3 presents the methodology in four parts: First, how to identify the minimum and maximum elements of a finite set using an MIP problem is presented. Next, the representation of a fuzzy multiplication as a set of inequalities, and the conversion from an FPFLES to an MIP problem are demonstrated. Finally, the step-by-step approach to solving an FPFLES is shown. Some numerical examples are discussed in Sect. 4, and the conclusion is given in Sect. 5. The list of abbreviations used in this study is presented in Table 1.

Table 1 List of abbreviations

Abbreviation	Description
FFLES	Fully Fuzzy Linear Equations System
FPFLES	Fully Pythagorean Fuzzy Linear Equations System
LES	Linear Equations System
MIP	Mixed Integer Programming
PFN	Pythagorean Fuzzy Number
PFS	Pythagorean Fuzzy Set

2 Preliminaries

This section reviews some necessary concepts to develop an FPFLES. First, a triangular PFN is defined, arithmetic operations on these numbers and definition of fuzzy equality are presented, and then an FPFLES is determined. Finally, the distance measure between triangular PFNs is defined, and the solution types are explained through the system using this distance measure.

Definition 1 [51] Let X be a fixed set. A PFS A in X is defined such as

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$$

where $\mu_A(\cdot)$ is the membership function and $\nu_A(\cdot)$ is the non-membership function expressed as $\mu_A, \nu_A : X \to [0, 1]$, $\forall x \in X$. There is a relaxation in the relation $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, and also, another relation is defined between the functions such that $0 \leq \mu_A^2(x) + \nu_A^2(x) \leq 1$ $\forall x \in X$. A Pythagorean fuzzy degree of hesitancy of x in A is expressed as $\pi_A(x) = \sqrt{1 - \mu_A^2(x) - \nu_A^2(x)}$, $\forall x \in X$. Therefore, $\beta = (\mu_\beta, \nu_\beta)$ is defined as a PFN by [53] where $\mu_\beta, \nu_\beta \in [0, 1]$, $\mu_\beta^2 + \nu_\beta^2 \leq 1$ and $\pi_\beta = \sqrt{1 - \mu_\beta^2 - \nu_\beta^2}$.

Definition 2 [33] A triangular PFN $A = \{(a, b, c); t, f\}$, which is a PFS defined on $\mathbb{R}$, is expressed as

$$\mu_A(x) = \begin{cases} \dfrac{(x-a)t}{b-a}, & a \leq x < b \\ t, & x = b \\ \dfrac{(c-x)t}{c-b}, & b < x \leq c \\ 0, & x < a \quad or \quad x > c \end{cases}$$

$$\nu_A(x) = \begin{cases} \dfrac{(b-x)+(x-a)f}{b-a}, & a \le x < b \\ f, & x = b \\ \dfrac{(x-b)+(c-x)f}{c-b}, & b < x \le c \\ 1, & x < a \quad or \quad x > c \end{cases}$$

where the values t and f correspond to the maximum degree of μ_A and the minimum degree of ν_A, respectively. These values are $t, f \in [0, 1]$ and $0 \le t^2 + f^2 \le 1$.

When the values are taken as $t = 1$ and $f = 0$, a triangular PFN takes a form $A = \{(a, b, c); (a', b, c')\}$ where membership function is

$$\mu_A(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x = b \\ \dfrac{c-x}{c-b}, & b < x \le c \\ 0, & x < a \quad or \quad x > c \end{cases}$$

and non-membership function is

$$\nu_A(x) = \begin{cases} \dfrac{(b-x)}{b-a'}, & a' \le x < b \\ 0, & x = b \\ \dfrac{(x-b)}{c'-b}, & b < x \le c' \\ 1, & x < a' \quad or \quad x > c' \end{cases}$$

where $a' \le a \le b \le c \le c'$.

Definition 3 [36] Let $A = \{(a, b, c); (a', b', c')\}$ and $B = \{(p, q, r); (p', q, r')\}$ be two triangular PFNs and $\alpha \in \mathbb{R}$. Some basic arithmetic operations can be defined as

$$\begin{aligned} A \oplus B &= \{(a+p, b+q, c+r); (a'+p', b+q, c'+r')\} \\ -A &= \{(-c, -b, -a); (-c', -b, -a')\} \\ A \ominus B &= \{(a-r, b-q, c-p); (a'-r', b-q, c'-p')\} \\ \alpha A &= \begin{cases} \{(\alpha a, \alpha b, \alpha c); (\alpha a', \alpha b, \alpha c')\}, & \alpha \ge 0 \\ \{(\alpha c, \alpha b, \alpha a); (\alpha c', \alpha b, \alpha a')\}, & \alpha < 0. \end{cases} \end{aligned}$$

Definition 4 [7] Let $A = \{(a, b, c); (a', b', c')\}$ be a triangular PFN. If $a' \ge 0$, then A is non-negative; if $a' \in \mathbb{R}$, then A is unrestricted (mixed).

Definition 5 Let $A = \{(a, b, c); (a', b', c')\}$ and $B = \{(p, q, r); (p', q, r')\}$ be two arbitrary triangular PFNs, then

$$\mu_{A \otimes B} = \begin{cases} (min\{ap, cp\}, bq, max\{ar, cr\}), & a \ge 0, \\ (min\{ar, cp\}, bq, max\{ap, cr\}), & a < 0 \; and \; c \ge 0, \\ (min\{ar, cr\}, bq, max\{ap, cp\}), & c < 0 \end{cases}$$

and

$$\nu_{A\otimes B} = \begin{cases} (min\{a'p', c'p'\}, bq, max\{a'r', c'r'\}), & a' \geq 0, \\ (min\{a'r', c'p'\}, bq, max\{a'p', c'r'\}), & a' < 0 \; and \; c' \geq 0, \\ (min\{a'r', c'r'\}, bq, max\{a'p', c'p'\}), & c' < 0. \end{cases}$$

Definition 6 Let $A = \{(a, b, c); (a', b, c')\}$ and $B = \{(p, q, r); (p', q, r')\}$ be two arbitrary triangular PFNs. They are said to be equal iff $a = p, b = q, c = r, a' = p'$, and $c' = r'$.

Definition 7 A FPFLES with triangular PFNs has the following form

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= B_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= B_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= B_m \end{aligned}$$

where $A_{ij} = \{(a_{ij}, b_{ij}, c_{ij}); (a'_{ij}, b_{ij}, c'_{ij})\}$ and $B_i = \{(b_i, g_i, h_i); (b'_i, g_i, h'_i)\}$ $(i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n)$ are arbitrary triangular PFNs, and $x_j = \{(x_j, y_j, z_j); (x'_j, y_j, z'_j)\}$ are Pythagorean fuzzy variables in triangular form. The FPFLES can be written as $\mathbb{A}X = \mathbb{B}$ in matrix form.

It is seen that the left-hand side of each equation of $\mathbb{A}X = \mathbb{B}$ contains the multiplication of PFNs and Pythagorean fuzzy variables. Thus, it is indivisible and therefore, insolvable. Moreover, since the components of the system $A_{ij}x_j$ are arbitrary, the solution process becomes more complicated.

Definition 8 Let $A = \{(a, b, c); (a', b, c')\}$ and $B = \{(p, q, r); (p', q, r')\}$ be two arbitrary triangular PFNs. The distance measure D between these triangular PFNs can be defined as the sum of the distances of membership values and non-membership values such that

$$D(A, B) = d\left((a, b, c), (p, q, r)\right) + d\left((a', b, c'), (p', q, r')\right)$$

where d is a distance measure for fuzzy numbers.

Theorem 1 *The distance measure D for arbitrary triangular PFNs A, B and C satisfy the following conditions:*

(i) $D(A, B) \geq 0$,
(ii) $D(A, B) = D(B, A)$,
(iii) $D(A, B) = 0$ *iff* $A = B$,
(iv) $D(A, C) \leq D(A, B) + D(B, C)$.

Proof For fuzzy numbers, the distance measure d provides the positivity, symmetry, equality, and triangular inequality properties. Furthermore, a PFN $A = \{(a, b, c); (a', b, c')\}$ is defined as having the maximum degree of membership degree of $\mu_A(x)$, i.e. $t = 1$, and the minimum degree of membership degree of $\nu_A(x)$, i.e. $f = 0$. The complement of A, A^c, has the values $t = 0$ and $f = 1$, which shows that the non-membership part of A can be determined as a triangular fuzzy number maintaining the order $a' \leq a \leq b \leq c \leq c'$.

(i) $d((a, b, c), (p, q, r)) \geq 0$ and $d\left((a', b, c'), (p', q, r')\right) \geq 0$ imply $D(A, B) \geq 0$.
(ii) Since $d((a, b, c), (p, q, r)) = d((p, q, r), (a, b, c))$ and $d\left((a', b, c'), (p', q, r')\right) = d\left((p', q, r'), (a', b, c')\right)$, D satisfies *(ii)*.
(iii) $(\rightarrow)$ If $D(A, B) = 0$, $d((a, b, c), (p, q, r)) = 0$ and $d\left((a', b, c'), (p', q, r')\right) = 0$ considering *(i)*. This implies the equality of A and B. If $A = B$, then Definition 6 is satisfied. $(\leftarrow)$ Because, $d((a, b, c), (p, q, r)) = 0$ and $d\left((a', b, c'), (p', q, r')\right) = 0$, $D(A, B) = 0$ is straightforward.
(iv) Since d satisfies the triangle inequality, the proof is obvious.

□

The closeness between $\mathbb{A}X$ and $\mathbb{B}$ is determined via the distance measure D for each equation individually, that is

$$\mathcal{D}(\mathbb{A}X, \mathbb{B}) = \sum_{i=1}^{m} D\left(\sum_{j=1}^{n} A_{ij}x_j, B_i\right) \tag{1}$$

The closeness determines whether the solution of the FPFLES is exact or approximate. In this paper, the types of solution are demonstrated considering the distance function given the study of Ebadi et al. [19]. To guarantee that the solution satisfies each FPFLES equation, the distance measure between the left-hand and right-hand sides is measured and then the sum of these distances is examined. To evaluate the resemblance between two fuzzy numbers, different distance measures can be used from the literature, such as [3, 8, 23, 31]. The choice of distance measure function does not affect the solution found by the proposed approach.

3 Methodology

The solution approach is a modified version of the study [27], and is based on identifying the minimum and maximum elements of a finite set through an MIP problem and transforming the multiplication of two arbitrary triangular PFNs into an MIP problem. This study combines these ideas with the fuzzy equality concept to solve an FPFLES. The primary purpose of the adapted approach is to determine the types of solutions to the FPFLES.

This section first presents how to identify the minimum and maximum elements of a finite set using an MIP problem, then explains how to transform a fuzzy multiplication into a set of inequalities, clarifies the transformation from an FPFLES to an MIP problem, and finally determines the type of a solution to the FPFLES.

3.1 Identifying the minimum and maximum elements of a finite set through a MIP problem

To present the fundamental basis of the approach, first, how to find the maximum and minimum elements of a finite set is demonstrated. Then, how to formulate an MIP problem to identify these maximum and minimum elements is shown.

Definition 9 [27] Consider a nonempty finite set F subset of the real numbers. The minimum element of F is found when $m_1 \in F$, and $m_1 \leq x$ for all $x \in F$. Similarly, the maximum element of F is found if $m_2 \in F$, and $x \leq m_2$ for all $x \in F$.

Theorem 2 *Let $F = \{f_1, f_2, ..., f_m\}$ be a nonempty finite subset of the set of real numbers. The set F has both a minimum and a maximum element. For proof, please check [18].*

To find the minimum and maximum values, introducing binary variables helps model the conditions for selecting one of these values as the minimum or maximum. Therefore, the following proposition is given.

Proposition 1 *Let $F = \{f_1, f_2, ..., f_m\}$ be a nonempty finite subset of the set of real numbers, $K = \{k_1, k_2, ..., k_m\}$ be a set of binary variables where $k_i = \{0, 1\}$ for all $i = 1, 2, ..., m$, and λ be a continuous variable representing the minimum element of the set. Each binary variable k_i indicates if f_i is the minimum. The following MIP problem*

$$Min\ 0.\lambda \tag{2a}$$

s.t.

$$\lambda \leq f_i \quad \forall i, \tag{2b}$$

$$f_i - \lambda \leq M(1 - k_i) \quad \forall i, \tag{2c}$$

$$\sum_{i=1}^{m} k_i = 1 \tag{2d}$$

yields finding the minimum element of the set F. Similarly, let β be a continuous variable representing the maximum element of the set F. The following MIP problem

$$Min\ 0.\beta \tag{3a}$$

s.t.

$$\beta \geq f_i \quad \forall i, \tag{3b}$$

$$\beta - f_i \leq M(1 - k_i) \quad \forall i, \tag{3c}$$

$$\sum_{i=1}^{m} k_i = 1, \tag{3d}$$

helps to determine the maximum element of the set F.

The proof of Proposition 1 can be checked from [27].

Here (2b) is the minimum condition, and (2c) is the selection constraint that forces λ to equal one of the f_i values using a large constant M. This constraint ensures that if $k_i = 1$, then $\lambda = f_i$. The last constraint (2d), guarantees that just one f_i is selected as the minimum. Since the aim is finding the minimum element, the objective function (2a) is set as a dummy objective to ensure the optimization solver runs properly. From the other side, (3b) is the maximum condition, (3c) is the selection constraint including a large constant M, and (3d) is the uniqueness of the maximum value. Because the maximum element is searched, the objective function (3a) is set as a dummy objective. Therefore, it can be said that the MIP problems (2) and (3) always have optimal solutions.

3.2 Transformation of a fuzzy multiplication to a set of inequalities

After establishing the minimum and maximum elements of a finite set via an MIP problem, its application in the multiplication of two arbitrary triangular PFN is presented. The multiplication of $A_{ij}x_j$ is determined, where $A_{ij} = \{(a_{ij}, b_{ij}, c_{ij}); (a'_{ij}, b_{ij}, c'_{ij})\}$ is a triangular PFN, and $x_j = \{(x_j, y_j, z_j); (x'_j, y_j, z'_j)\}$ is a Pythagorean fuzzy variable. The multiplication of two arbitrary triangular PFN can be defined using the extension principle given in Definition 5 such that

$$A_{ij}x_j = \{(\lambda_{ij}, \alpha_{ij}, \beta_{ij}); (\lambda'_{ij}, \alpha_{ij}, \beta'_{ij})\} \tag{4}$$

where

$$\begin{aligned} \lambda_{ij} &= min\{a_{ij}x_j, c_{ij}x_j\} \\ \alpha_{ij} &= b_{ij}y_j \\ \beta_{ij} &= max\{a_{ij}z_j, c_{ij}z_j\} \\ \lambda'_{ij} &= min\{a'_{ij}x'_j, c'_{ij}x'_j\} \\ \beta'_{ij} &= max\{a'_{ij}z'_j, c'_{ij}z'_j\}. \end{aligned} \tag{5}$$

Here, the values of λ_{ij} and λ'_{ij} are identified according to the sign of the components A_{ij}: A_{ij} can be positive, unrestricted, or negative depending on whether it belongs to one of the following sets

$$\begin{aligned} S^{pos} &= \left\{(i, j) : a_{ij} \geq 0\right\} \\ S^{mix} &= \left\{(i, j) : a_{ij} \leq 0 \; and \; c_{ij} \geq 0\right\} \\ S^{neg} &= \left\{(i, j) : c_{ij} < 0\right\} \end{aligned} \tag{6}$$

where (i, j) indicates the position of A_{ij} in the FPFLES. Thus, the multiplication of $A_{ij}x_j$ can be determined considering (6) as follows

$$\begin{aligned} A_{ij}x_j = \{&\left(\min\left\{a_{ij}x_j, c_{ij}x_j\right\}, b_{ij}y_j, \max\left\{a_{ij}z_j, c_{ij}z_j\right\}\right); \\ &(\min\left\{a'_{ij}x'_j, c'_{ij}x'_j\right\}, b_{ij}y_j, \max\left\{a'_{ij}z'_j, c_{ij}z'_j\right\})\} \end{aligned} \quad \forall (i, j) \in S^{pos}, \tag{7}$$

$$A_{ij}x_j = \{\left(\min\left\{a_{ij}z_j, c_{ij}x_j\right\}, b_{ij}y_j, \max\left\{a_{ij}x_j, c_{ij}z_j\right\}\right); \quad \forall (i,j) \in S^{mix}, \\ (\min\left\{a'_{ij}z'_j, c'_{ij}x'_j\right\}, b_{ij}y_j, \max\left\{a'_{ij}x'_j, c'_{ij}z'_j\right\})\} \tag{8}$$

$$A_{ij}x_j = \{\left(\min\left\{a_{ij}z_j, c_{ij}z_j\right\}, b_{ij}y_j, \max\left\{a_{ij}x_j, c_{ij}x_j\right\}\right); \quad \forall (i,j) \in S^{neg}. \\ (\min\left\{a'_{ij}z'_j, c'_{ij}z'_j\right\}, b_{ij}y_j, \max\left\{a'_{ij}x'_j, c'_{ij}x'_j\right\})\} \tag{9}$$

A non-linearity situation occurs since the components on the right-hand side of the FPFLES in Definition 7 are not constant. To eliminate this non-linearity, Proposition 1 is applied to select the minimum and maximum component in each (7), (8), and (9). To show this implementation, the multiplication defined in (7) is considered. For an $(i, j) \in S^{pos}$, the left component $\lambda_{ij} = \min\left\{a_{ij}x_j, c_{ij}x_j\right\}$ of (7) can be transformed into the following set of inequalities

$$\begin{aligned} \lambda_{ij} &\le a_{ij}x_j \\ \lambda_{ij} &\le c_{ij}x_j \\ a_{ij}x_j - \lambda_{ij} &\le Mk_{ijL} \\ c_{ij}x_j - \lambda_{ij} &\le M\left(1 - k_{ijL}\right) \end{aligned} \tag{10}$$

where M is a large constant, and k_{ijL} is a binary variable. The utilization of a binary variable k_{ijL} facilitates the identification of the minimum value between $a_{ij}x_j$ and $c_{ij}x_j$ by formulating inequalities that result in a feasible solution depending on the value of the binary variable. When the binary variable is 0, it indicates that the minimum value should be $a_{ij}x_j$; otherwise, the minimum value should be $c_{ij}x_j$. The essence of this approach involves employing an exceedingly large value M to convert the inequalities derived from $a_{ij}x_j$ and $c_{ij}x_j$. The inequalities are constructed in such a way that when $k_{ijL} = 0$, only $k_{ijL} = a_{ij}x_j$ fulfills the inequalities; otherwise, $k_{ijL} = c_{ij}x_j$. This approach enables us the selection of the minimum value between two elements depending on the value of a binary variable without employing a conventional minimum operator. Additionally, it guarantees that k_{ijL} will assume the value of the minimum between $a_{ij}x_j$ and $c_{ij}x_j$.

The right component in the membership part of (7), i.e. $\beta_{ij} = \max\left\{a_{ij}z_j, c_{ij}z_j\right\}$, can be written as

$$\begin{aligned} \beta_{ij} &\ge a_{ij}z_j \\ \beta_{ij} &\ge c_{ij}z_j \\ \beta_{ij} - a_{ij}z_j &\le Mk_{ijR} \\ \beta_{ij} - c_{ij}z_j &\le M\left(1 - k_{ijR}\right) \end{aligned} \tag{11}$$

where $M >> 0$ and $k_{ijR} \in \{0, 1\}$. As it is seen that the max operator in β_{ij} variable contains two elements, i.e. $a_{ij}z_j$ and $c_{ij}z_j$. This operator generates one of these two elements. Thus, this situation turns into a problem of choosing one of these elements. This is achieved through binary variables so that a MIP problem is constructed.

Similarly, for an $(i, j) \in S^{pos}$, the left component $\lambda'_{ij} = \min\left\{a'_{ij}x'_j, c'_{ij}x'_j\right\}$ of (7) can be rewritten as

$$\begin{aligned}\lambda'_{ij} &\leq a'_{ij}x'_j\\ \lambda'_{ij} &\leq c'_{ij}x'_j\\ a'_{ij}x'_j - \lambda'_{ij} &\leq Mk'_{ijL}\\ c'_{ij}x'_j - \lambda'_{ij} &\leq M\left(1 - k'_{ijL}\right)\end{aligned} \tag{12}$$

and the right component $\beta'_{ij} = \max\left\{a'_{ij}z'_j, c'_{ij}z'_j\right\}$ can be rewritten as

$$\begin{aligned}\beta'_{ij} &\geq a'_{ij}z'_j\\ \beta'_{ij} &\geq c'_{ij}z'_j\\ \beta'_{ij} - a'_{ij}z'_j &\leq Mk'_{ijR}\\ \beta'_{ij} - c'_{ij}z'_j &\leq M\left(1 - k'_{ijR}\right)\end{aligned} \tag{13}$$

where M is a large constant, and $k'_{ijL}, k'_{ijR} \in \{0, 1\}$ are binary variables.

It is seen that the min operator in λ_{ij} variable, which is in the membership part, contains two arguments such that $a_{ij}x_j$ or $c_{ij}x_j$. This operator generates one of these two arguments. Therefore, choosing one of these arguments becomes a problem for finding a feasible solution for the FPFLES. This can be determined as a MIP problem constructed using binary variables. This process is valid for the max operator in the β_{ij} variable. These variables are defined for the membership part, and λ'_{ij} and β'_{ij} are defined for the non-membership part in the min and max operators, respectively. The solution approach describes how to obtain the $\lambda_{ij}, \lambda'_{ij}, \beta_{ij}$ and β'_{ij} for an $(i, j) \in S^{pos}$. It can also be applied for all (i, j) in S^{neg} and S^{mix}.

3.3 Conversion of an FPFLES to an MIP problem

Up to this subsection, this paper has demonstrated how to obtain the minimum and maximum element of a finite set using an MIP problem and how to apply this transformation to components of the multiplication of two triangular PFNs contained on the right-hand side of an FPFLES. This subsection shows how to construct an MIP problem equivalent to an FPFLES.

Consider an FPFLES given in Definition 7. This system contains the fuzzy multiplications of $A_{ij}x_j$ on the left-hand side. According to the sign of A_{ij}, one of the fuzzy multiplications (7), (8), or (9) is determined, and the fuzzy multiplication is transformed into the inequalities (10), (11), (12), and (13). Every term in each equation in FPFLES undergoes this process.

Since the fuzzy multiplication $A_{ij}x_j$ is determined in (4) and the right-hand side is B_i, the FPFLES is rewritten as

$$\sum_{j=1}^{n}\{(\lambda_{ij}, \alpha_{ij}, \beta_{ij}); (\lambda'_{ij}, \alpha_{ij}, \beta'_{ij})\} = \{(b_i, g_i, h_i); (b'_i, g_i, h'_i)\} \quad \forall i. \tag{14}$$

Using the fuzzy equality given in Definition 6, (14) can be converted into the following equations

$$\sum_{j=1}^{n} \lambda_{ij} = b_i \tag{15a}$$

$$\sum_{j=1}^{n} \alpha_{ij} = g_i \tag{15b}$$

$$\sum_{j=1}^{n} \beta_{ij} = h_i \tag{15c}$$

$$\sum_{j=1}^{n} \lambda'_{ij} = b'_i \tag{15d}$$

$$\sum_{j=1}^{n} \beta'_{ij} = h'_i. \tag{15e}$$

for all $i = 1, 2, \ldots, m$. Since the equations system (15) and the FPFLES are equivalent, they have identical solutions. However, the variables λ_{ij}, β_{ij}, λ'_{ij}, and β'_{ij} contain nonlinearity. Therefore, they are determined using the set of inequalities in (10), (11), (12), and (13), respectively and the nonlinearity is eliminated.

Since the solution approach is based on the sign of A_{ij}, the coefficient matrix can be adapted to the FPFLES by defining the following function

$$\varepsilon(A_{ij}) = \begin{cases} [1\ 0\ 0], & (i, j) \in S^{pos}, \\ [0\ 1\ 0], & (i, j) \in S^{mix}, \\ [0\ 0\ 1], & (i, j) \in S^{neg} \end{cases} \tag{16}$$

The MIP problem equivalent to (15) can be written as follows

$$Min\ obj = \sum_{i=1}^{m}\sum_{j=1}^{n} 0.\lambda_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} 0.\beta_{ij} \tag{17a}$$

s.t.

$$\left.\begin{array}{l}\varepsilon(A_{ij})\ \left[\lambda_{ij}-a_{ij}x_j\quad \lambda_{ij}-a_{ij}z_j\quad \lambda_{ij}-a_{ij}z_j\right]^T\le 0\\ \varepsilon(A_{ij})\ \left[\lambda_{ij}-c_{ij}x_j\quad \lambda_{ij}-c_{ij}x_j\quad \lambda_{ij}-c_{ij}z_j\right]^T\le 0\\ \varepsilon(A_{ij})\ \left[a_{ij}x_j-\lambda_{ij}-Mk_{ijL}\quad a_{ij}z_j-\lambda_{ij}-Mk_{ijL}\quad a_{ij}z_j-\lambda_{ij}-Mk_{ijL}\right]^T\le 0\\ \varepsilon(A_{ij})\ \left[c_{ij}x_j-\lambda_{ij}-Mk_{ijL}\quad c_{ij}x_j-\lambda_{ij}-Mk_{ijL}\quad c_{ij}z_j-\lambda_{ij}-Mk_{ijL}\right]^T\le 0\end{array}\right\}\forall(i,j) \tag{17b}$$

$$\left.\begin{array}{l}\varepsilon(A_{ij})\ \left[\beta_{ij}-a_{ij}z_j\quad \beta_{ij}-a_{ij}x_j\quad \beta_{ij}-a_{ij}x_j\right]^T\ge 0\\ \varepsilon(A_{ij})\ \left[\beta_{ij}-c_{ij}z_j\quad \beta_{ij}-c_{ij}z_j\quad \beta_{ij}-c_{ij}x_j\right]^T\ge 0\\ \varepsilon(A_{ij})\ \left[\beta_{ij}-a_{ij}z_j-Mk_{ijR}\quad \beta_{ij}-a_{ij}x_j-Mk_{ijR}\quad \beta_{ij}-a_{ij}x_j-Mk_{ijR}\right]^T\le 0\\ \varepsilon(A_{ij})\ \left[\beta_{ij}-c_{ij}z_j-M(1-k_{ijR})\quad \beta_{ij}-c_{ij}z_j\right.\\ \qquad \left.-M(1-k_{ijR})\quad \beta_{ij}-c_{ij}x_j-M(1-k_{ijR})\right]^T\le 0\end{array}\right\}\forall(i,j) \tag{17c}$$

$$\left.\begin{array}{l}\varepsilon(A_{ij})\ \left[\lambda'_{ij}-a'_{ij}x'_j\quad \lambda'_{ij}-a'_{ij}z'_j\quad \lambda'_{ij}-a'_{ij}z'_j\right]^T\le 0\\ \varepsilon(A_{ij})\ \left[\lambda'_{ij}-c'_{ij}x'_j\quad \lambda'_{ij}-c'_{ij}x'_j\quad \lambda'_{ij}-c'_{ij}z'_j\right]^T\le 0\\ \varepsilon(A_{ij})\ \left[a'_{ij}x'_j-\lambda'_{ij}-Mk'_{ijL}\quad a'_{ij}z'_j-\lambda'_{ij}-Mk'_{ijL}\quad a'_{ij}z'_j-\lambda'_{ij}-Mk'_{ijL}\right]^T\le 0\\ \varepsilon(A_{ij})\ \left[c_{ij}x_j-\lambda_{ij}-Mk_{ijL}\quad c_{ij}x_j-\lambda_{ij}-Mk_{ijL}\quad c_{ij}z_j-\lambda_{ij}-Mk_{ijL}\right]^T\le 0\end{array}\right\}\forall(i,j) \tag{17d}$$

$$\left.\begin{array}{l}\varepsilon(A_{ij})\ \left[\beta'_{ij}-a'_{ij}z'_j\quad \beta'_{ij}-a'_{ij}x'_j\quad \beta'_{ij}-a'_{ij}x'_j\right]^T\ge 0\\ \varepsilon(A_{ij})\ \left[\beta'_{ij}-c'_{ij}z'_j\quad \beta'_{ij}-c'_{ij}z'_j\quad \beta'_{ij}-c'_{ij}x'_j\right]^T\ge 0\\ \varepsilon(A_{ij})\ \left[\beta'_{ij}-a'_{ij}z'_j-Mk'_{ijR}\quad \beta'_{ij}-a'_{ij}x'_j-Mk'_{ijR}\quad \beta'_{ij}-a'_{ij}x'_j-Mk'_{ijR}\right]^T\le 0\\ \varepsilon(A_{ij})\ \left[\beta'_{ij}-c'_{ij}z'_j-M(1-k'_{ijR})\quad \beta'_{ij}-c'_{ij}z'_j\right.\\ \qquad \left.-M(1-k'_{ijR})\quad \beta'_{ij}-c'_{ij}x'_j-M(1-k'_{ijR})\right]^T\le 0\end{array}\right\}\forall(i,j) \tag{17e}$$

$$\left.\begin{array}{l}\sum_{j=1}^{n}\lambda_{ij}=b_i\\ \sum_{j=1}^{n}\alpha_{ij}=g_i\\ \sum_{j=1}^{n}\beta_{ij}=h_i\\ \sum_{j=1}^{n}\lambda'_{ij}=b'_i\\ \sum_{j=1}^{n}\beta'_{ij}=h'_i\end{array}\right\}\forall i \tag{17f}$$

$$\left.\begin{array}{l}y_j-x_j\ge 0\\ z_j-y_j\ge 0\end{array}\right\}\forall j \tag{17g}$$

$$\left.\begin{array}{l}x_j-x'_j\ge 0\\ z'_j-z_j\ge 0\end{array}\right\}\forall j \tag{17h}$$

$$\begin{array}{l}\lambda_{ij},\lambda'_{ij},\alpha_{ij},\beta_{ij},\beta'_{ij}\quad free\quad \forall(i,j)\\ x_j,y_j,z_j,x'_j,z'_j\quad free\quad \forall j\\ k_{ijL},k'_{ijL},k_{ijR},k'_{ijR}\in\{0,1\},\quad \forall(i,j)\end{array} \tag{17i}$$

where $[]^T$ is the transpose of the row matrix. In the MIP model, the constraints (17b)-(17e) select the minimum and maximum values contained in each fuzzy multiplication; (17f) satisfy the equalities between the left-hand and right-hand sides of FPFLES; (17g) maintain the order between the components of a triangular PFN in the membership part; (17h) satisfy the order among the components of a triangular PFN; and (17i) are the sign constraints.

Theorem 3 *The exact solution of the FPFLES is the optimal solution of the MIP problem (17).*

Proof Considering the subsection 3.1 and equalities (15), the proof is straightforward. □

Finding the optimal solution to the MIP problem determines the type of solution, either exact or approximate, to the FPFLES. However, if the MIP problem lacks a feasible solution, the FPFLES is considered to have no solution. Therefore, the following definitions are given to determine the types of solutions to the FPFLES:

Definition 10 For all j, $\mathbf{x_j} = \{(\mathbf{x_j}, \mathbf{y_j}, \mathbf{z_j}); (\mathbf{x'_j}, \mathbf{y_j}, \mathbf{z'_j})\}$ is an exact Pythagorean fuzzy solution of FPFLES if it solves (7).

Definition 11 Let a Pythagorean fuzzy number $A_{ij}x_j$ is denoted by $\{(\lambda_{ij}, \alpha_{ij}, \beta_{ij}); (\lambda'_{ij}, \alpha_{ij}, \beta'_{ij})\}$ and $B_i = \{(b_i, g_i, h_i); (b'_i, g_i, h'_i)\}$. For all j, $\mathbf{x_j} = \{(\mathbf{x_j}, \mathbf{y_j}, \mathbf{z_j}); (x'_j, y_j, z'_j)\}$ is said to be a restricted approximate Pythagorean fuzzy solution to the FPFLES if it satisfies the following conditions:

- For all j, $\mathbf{x_j}$ is not a feasible Pythagorean fuzzy solution,
- $b_i \leq \lambda_{ij}$ and $h_i \geq \beta_{ij}$, and $b'_i \leq \lambda'_{ij}$ and $h'_i \geq \beta'_{ij}$ for all i.

Definition 12 Let a Pythagorean fuzzy number $A_{ij}x_j$ is denoted by $\{(\lambda_{ij}, \alpha_{ij}, \beta_{ij}); (\lambda'_{ij}, \alpha_{ij}, \beta'_{ij})\}$ and $B_i = \{(b_i, g_i, h_i); (b'_i, g_i, h'_i)\}$. For all j, $\mathbf{x_j} = \{(\mathbf{x_j}, \mathbf{y_j}, \mathbf{z_j}); (x'_j, y_j, z'_j)\}$ is said to be a relaxed approximate Pythagorean fuzzy solution to the FPFLES if it satisfies the following conditions:

- For all j, $\mathbf{x_j}$ is not a feasible Pythagorean fuzzy solution,
- $b_i \leq \beta_{ij}$ and $h_i \geq \lambda_{ij}$, and $b'_i \leq \beta'_{ij}$ and $h'_i \geq \lambda'_{ij}$ for all i.

Little difference can be seen between Definition 11 and Definition 12. In Definition 11, when the solution obtained is substituted in an equation of the FPFLES, the support of the LHS value is covered by the support of the RHS value, and this is provided for each equation. On the other hand, Definition 12 states that the support of LHS value obtained from the substitution can be covered by the support of RHS value or not.

In addition to these definitions, the types of solutions can be determined using the distance measure presented in (1). In this study, the distance measure function proposed by Ebadi et al. [19] is used. Despite different values being obtained with the different distance measure functions, the determination of exact and approximate solutions will remain same. Accordingly, if $\mathcal{D}(\mathbb{A}X, \mathbb{B}) = 0$ for the solution $\mathbf{x_j}$, then the $\mathbf{x_j}$ is an exact Pythagorean fuzzy solution to the FPFLES. If $\mathcal{D}(\mathbb{A}X, \mathbb{B}) \neq 0$, then $\mathbf{x_j}$ is an approximate fuzzy Pythagorean solution to the FPFLES.

The solution types of FPFLES can be represented in Fig. 1.

Figure 1 illustrates the solution cases for any equation in the FPFLES, specifically for membership function values of a Pythagorean fuzzy number. In Figure 1a, the

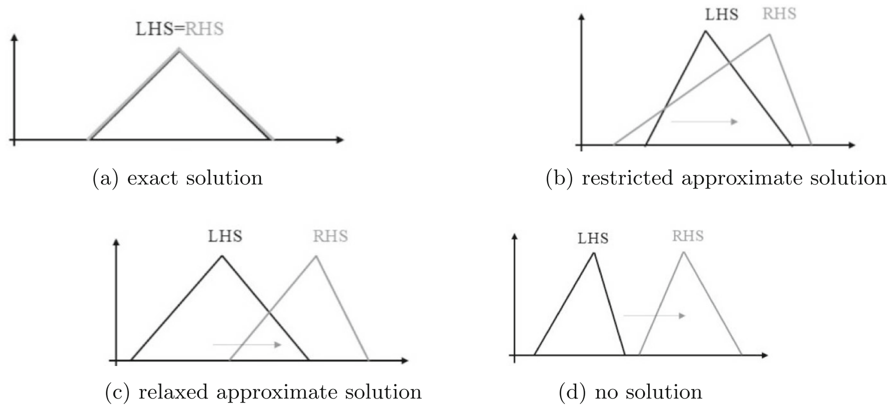

Fig. 1 Solution types for a FPFLES

LHS is equal to the RHS, which means the solution is exact. It should be noted that in order for the solution found to be an exact solution of the FPFLES, the structure given in Figure 1a must be satisfied for all equations. In Figure 1b, the distance measure between the LHS and RHS is different than zero, which is an approximate solution. The classification of this approximate solution is restricted because the support of the RHS value covers the support of the LHS value. Figure 1c illustrates the relaxed approximate solution, as the intersection of the LHS and RHS support sets is not null. In Figure 1d, there are no mutual points for both LHS and RHS, which means there is no solution. Figure 1 can also be illustrated for non-membership function values of the Pythagorean fuzzy number similarly.

3.4 Determination of the type of solution to FPFLES

The following steps can be applied to solve an FPFLES:

Step 1 Consider an FPFLES having Pythagorean fuzzy parameters A_{ij} in triangular form.

Step 2 For all (i, j) pairs $i = 1, 2, \dots, m;\ \ j = 1, 2, \dots, n$:

- (i) Determine the set to which the parameter A_{ij} belongs: S^{pos}, S^{mix}, or S^{neg}.
- (ii) Construct the components defined in (5) such as λ_{ij}, λ'_{ij}, β_{ij}, and β'_{ij}, and define binary variables k_{ijL}, k'_{ijL}, k_{ijR}, and k'_{ijR} corresponding to these components.
- (iii) Construct the set of inequalities as given in (10) for the component λ_{ij}, and similarly, apply the same process as in (11), (12), and (13) for the components λ'_{ij}, β_{ij}, and β'_{ij}, respectively. Add all these inequalities into an MIP problem as constraints.

Step 3 Write the fuzzy equalities given in (15) for all i, and include these equations into the MIP problem as constraints.

Step 4 Add the inequalities $y_j - x_j \geq 0$ and $z_j - y_j \geq 0$ for maintaining the order of the components in the membership part; and $x_j - x'_j \geq 0$ and $z'_j - z_j \geq 0$

for maintaining the order among the components of a triangular PFN for all j to the MIP problem as constraints.

Step 5 Solve the MIP problem (17) and analyze the solution as follows:

(i) If $obj = 0$, then the MIP problem has an optimal solution, and it is an exact Pythagorean fuzzy solution to the FPFLES. Go to Step 9.
(ii) If the MIP problem has alternative solutions, then the FPFLES has alternative Pythagorean fuzzy solutions. Go to Step 9.
(iii) If the MIP problem no feasible solution, then the FPFLES may have an approximate Pythagorean fuzzy solution. Continue.

Here, $\mathcal{D}(\mathbb{A}X, \mathbb{B}) = 0$ is obtained for (i) and (ii).

Step 6 For all $i = 1, 2, \ldots, m$:

(i) Rearrange the equality constraint (17f) by introducing Pythagorean fuzzy variables $L_i = \{(L_i^L, L_i^M, L_i^R); (L_i^{'L}, L_i^M, L_i^{'R})\}$ to the left-hand side and $R_i = \{(R_i^L, R_i^M, R_i^R); (R_i^{'L}, R_i^M, R_i^{'R})\}$ to the right-hand side of the constraints and rewrite the equations as follows:

$$\sum_{j=1}^{n} \lambda_{ij} + L_i^L = b_i + R_i^L$$

$$\sum_{j=1}^{n} \alpha_{ij} + L_i^M = g_i + R_i^M$$

$$\sum_{j=1}^{n} \beta_{ij} + L_i^R = h_i + R_i^R$$

$$\sum_{j=1}^{n} \lambda'_{ij} + L_i^{'L} = b'_i + R_i^{'L}$$

$$\sum_{j=1}^{n} \beta'_{ij} + L_i^{'R} = h'_i + R_i^{'R}$$

(ii) Rearrange the objective function (17a) consisting of the sum of the defined Pythagorean fuzzy variables:

$$Min\ obj = L_i^L + L_i^M + L_i^R + L_i^{'L} + L_i^{'R} + R_i^L + R_i^M + R_i^R + R_i^{'L} + R_i^{'R}.$$

This means that a modification is made to the objective function of the MIP problem to find a solution that minimizes the total error added to both sides of the equation.

(iii) Add new constraints to cover the LHS values:

$$\sum_{j=1}^{n} \lambda_{ij} - b_i \geq 0,$$

$$h_i - \sum_{j=1}^{n} \beta_{ij} \geq 0,$$

$$\sum_{j=1}^{n} \lambda'_{ij} - b'_i \geq 0,$$

$$h'_i - \sum_{j=1}^{n} \beta'_{ij} \geq 0.$$

Step 7 Solve the restricted MIP problem and analyze the solution as follows:

(i) If the restricted MIP problem has an optimal solution, then the FPFLES has a restricted approximate Pythagorean fuzzy solution. Go to Step 9.
(ii) Even if a restricted approximate solution has been found, it is possible to find a relaxed approximate solution. Continue.
(ii) If the restricted MIP problem has no feasible solution, then the FPFLES may have a relaxed approximate solution. Continue.

Step 8 Eliminate the particular block of constraints in Step 6 (iii) and obtain a relaxed MIP problem. Analyze the solution as follows:

(i) If the relaxed MIP problem has an optimal solution, then the FPFLES has a related approximate Pythagorean fuzzy solution. Go to Step 9.
(ii) If the relaxed MIP problem has no feasible solution, then it means there is no solution for the FPFLES. Stop.

Step 9 Determine the solution to the FPFLES. Stop.

4 Numerical examples

In this section, some numerical examples are illustrated to show the applicability of the solution approach to the FPFLES. The closeness between the LHS and RHS of each FPFLES helps to determine the type of solution to the FPFLES.

Example 1 Consider the following FPFLES:

$$\{(4, 7, 11); (2, 7, 13)\}x_1 \oplus \{(5, 8, 10); (3, 8, 12)\}x_2 = \{(23, 83, 157); (8, 83, 249)\}$$
$$\{(3, 5, 8); (1, 5, 11)\}x_1 \oplus \{(4, 7, 10); (2, 7, 13)\}x_2 = \{(18, 67, 136); (5, 67, 242)\}$$

where x_1, x_2 are triangular PFNs.

The adapted approach starts with multiplying the Pythagorean fuzzy parameters and variables. Consider multiplying A_{11} and x_1. Since the triangular PFN has membership and non-membership values, the multiplication is between $\{(4, 7, 11); (2, 7, 13)\}$

and $\{(x_1, y_1, z_1); (x_1', y_1, z_1')\}$. It is seen that $A_{11} \in S^{pos}$ which shows that the multiplication defined in (7) is applied. Hence, the multiplication of A_{11} and x_1 is such that

$$A_{11}x_1 = \{(\lambda_{11}, \alpha_{11}, \beta_{11}); (\lambda_{11}', \alpha_{11}, \beta_{11}')\}$$

where

$$\begin{aligned}\lambda_{11} &= \min\{4x_1, 11x_1\}\\ \alpha_{11} &= 7y_1\\ \beta_{11} &= \max\{4z_1, 11z_1\}\\ \lambda_{11}' &= \min\{2x_1', 13x_1'\}\\ \beta_{11}' &= \max\{2z_1', c_1'3z_1'\}.\end{aligned}$$

Here, the min and max operators yield a non-linearity situation. Thus, the situation is handled by converting these components into inequalities. To illustrate, only the first component λ_{11} is transformed into the following inequalities:

$$\begin{aligned}\lambda_{11} &\le 4x_1\\ \lambda_{11} &\le 11x_1\\ 4x_1 - \lambda_{11} &\le Mk_{11L}\\ 11x_1 - \lambda_{11} &\le M(1 - k_{11L})\end{aligned}$$

where M is a large constant, and k_{11L} is a binary variable. The same transformation is applied for all terms on the right-hand side of the FPFLES.

In the FPFLES, the parameters and the right-hand side values belong to the set S^{pos}. Hence, the same inequality forms given in (7) are used for the multiplications.

It is seen that the multiplication of A_{11} and x_1 yields 16 inequalities. Accordingly, there are 2 different multiplications in the first equation of the FPFLES: A_{11} and x_1, and A_{12} and x_2. Therefore, the FPFLES with the dimension 2×2 is converted into an MIP problem having $2\times2\times16$ inequalities. Moreover, the inequalities given in Step 4 are added to the MIP problem to maintain the order of a fuzzy variable's components. By solving the MIP problem, an optimal solution is found. Therefore, the exact Pythagorean fuzzy solution is found to be $x_1 = \{(2, 5, 7); (1, 5, 9)\}$ and $x_2 = \{(3, 6, 8); (2, 6, 11)\}$.

By substituting the exact Pythagorean fuzzy solution into each equation, the closeness between the RHS and LHS is found to be 0, which indicates the accuracy of the solution.

Since there is no research on FFLES with triangular PFNs in the literature, this example utilizing the equations from the LP problem with equality constraints in the article [7] is constructed.

Example 2 Consider the following FPFLES modified from an example in [4] by taking the triangular fuzzy numbers as the membership value of the triangular Pythagorean fuzzy numbers:

$$\{(3, 8, 12); (1, 8, 15)\}x_1 \oplus \{(-4, -3, 0); (-10, -3, 5)\}x_2$$
$$\oplus\{(-2, 3, 7); (-5, 3, 10)\}x_3 = \{(-43, 12, 104); (-100, 12, 205)\}$$
$$\{(-6, 0, 1); (-10, 0, 5)\}x_1 \oplus \{(-20, -10, 2); (-25, -10, 5)\}x_2$$

Table 2 LHS and RHS values and their distance measures for each equation in the case of restricted approximate solution

	LHS	RHS	Distance
Eq1	{(−14.667,−0.667,12.667);(−20,−0.667,20)}	{(−15,−2,15);(−20,−2,20)}	4.999
Eq2	{(−15,4,17.333);(−19.75,4,22)}	{(−15,4,20);(−25,4,24)}	7.92

$$\begin{aligned} &\oplus\{(-13,-6,-1);(-15,-6,5)\}x_3=\{(-61,24,125);(-225,24,225)\}\\ &\{(-30,-27,-21);(-35,-27,-15)\}x_1\oplus\{(5,6,7);(1,6,10)\}x_2\\ &\oplus\{(12,19,22);(10,19,25)\}x_3=\{(-356,-157,26);(-520,-157,130)\}\end{aligned}$$

where x_1, x_2 and x_3 are triangular PFNs.

Here, there are three different types of parameters belonging to the sets S^{pos}, S^{mix}, and S^{neg}. For example, $A_{11}\in S^{pos}$, $A_{22}\in S^{mix}$, and $A_{31}\in S^{neg}$, therefore, the multiplications defined in (7), (8), and (9) are applied, respectively.

Since the FPFLES has a 3×3 dimension, it is converted into an MIP problem with 3×3×16 inequalities. Then, the inequalities maintaining the order of a fuzzy variable's components, as given in Step 4, are added to the MIP problem. After solving the MIP problem, the exact Pythagorean fuzzy solution is found to be $x_1=\{(1,3,7);(0,3,9)\}$, $x_2=\{(-2,0,1);(-3,0,3)\}$, and $x_3=\{(-6,-4,-1);(-7,-4,4)\}$, and the closeness between LHS and RHS is equal to 0, accordingly.

Example 3 Consider the following FPFLES modified from an example in [4] by taking the triangular fuzzy numbers as the membership value of the triangular Pythagorean fuzzy numbers:

$$\begin{aligned} &\{(-3,-1,2);(-5,-1,3)\}x_1\oplus\{(1,2,4);(0,2,5)\}x_2\\ &\quad=\{(-15,-2,15);(-20,-2,20)\}\\ &\{(-4,-3,-2);(-5,-3,0)\}x_1\oplus\{(1,3,6);(0,3,8)\}x_2\\ &\quad=\{(-15,4,20);(-25,4,24)\}\end{aligned}$$

where x_1, x_2 are triangular PFNs.

Firstly, it should be determined whether there is an exact Pythagorean fuzzy solution for Example 3. Applying Step 1 through Step 5, an MIP problem is constructed and an infeasible solution is found. That is, there is no exact Pythagorean fuzzy solution to the FPFLES. Thus, to find an approximate Pythagorean fuzzy solution, the remaining steps are applied. When Steps 6 and 7 are processed, a restricted approximate Pythagorean fuzzy solution is found to be $x_1=\{(-3.333,-3,333,0,75);(-3.333,-3.333,1.8)\}$, $x_2=\{(-2,-2,0.667);(-2,-2,0.667)\}$. The solutions are substituted into the equations, the LHS values are calculated, and each distance measure between the LHS and RHS is found in Table 2.

Table 2 demonstrates that the closeness between the LHS and RHS is $\mathcal{D}(\mathbb{A}X,\mathbb{B})=12.919$.

Table 3 LHS and RHS values and their distance measures for each equation in the case of relaxed approximate solution

	LHS	RHS	Distance
Eq1	{(−14.667,−0.667,14.444);(−20,−0.667,22.222)}	{(−15,−2,15);(−20,−2,20)}	5.25
Eq2	{(−15,4,20);(−19.75,4,25.556)}	{(−15,4,20);(−25,4,24)}	3.226

By cleaning the constraints presented in Step 6 (iii), a relaxed MIP problem is obtained. After finding the optimal solution to the MIP problem, the relaxed approximate Pythagorean fuzzy solution is found as $x_1 = \{(-3.333, -3, 333, 0, 75); (-3.333, -3.333, 1.8)\}$, $x_2 = \{(-2, -2, 1.111); (-2, -2, 1.111)\}$. Table 3 presents each LHS value when the solution found is substituted into the equations and each corresponding RHS value and their distance measures.

The closeness is found to be $\mathcal{D}(\mathbb{A}X, \mathbb{B}) = 8.476$. It is clear that when the MIP problem is restricted, the distance measure value becomes higher than the measure obtained from the relaxed MIP problem.

Example 4 Consider the following FPFLES:

$$\begin{aligned}
&\{(2,3,5);(1,3,6)\}x_1 \oplus \{(1,2,2);(0,2,3)\}x_2\\
&\quad\oplus\{(-3,0,1);(-5,0,2)\}x_3 = \{(-7,4,20);(-20,4,50)\}\\
&\{(-3,2,5);(-4,2,6)\}x_1 \oplus \{(4,5,6);(3,5,7)\}x_2\\
&\quad\oplus\{(0,2,4);(-3,2,5)\}x_3 = \{(-13,6,23);(-32,6,58)\}\\
&\{(-3,-2,-1);(-5,-2,-1)\}x_1 \oplus \{(0,2,5);(-2,2,6)\}x_2\\
&\quad\oplus\{(-5,0,5);(-6,0,6)\}x_3 = \{(-18,4,33);(-47,4,58)\}\\
&\{(-1,0,1);(-2,0,2)\}x_1 \oplus \{(0,2,4);(-2,2,4)\}x_2\\
&\quad\oplus\{(4,5,6);(3,5,8)\}x_3 = \{(-19,-6,9);(-46,-6,22)\}
\end{aligned}$$

where x_1, x_2 and x_3 are triangular PFNs.

This synthetic example is constructed to show that the adapted approach can solve a non-square structured FPFLESs. Since the dimension of FPFLES is 4×3, an MIP problem has $4 \times 3 \times 16$ inequalities from the transformation process; also considering the equality constraints in Step 3 and inequality constraints in Step 4, the optimal solution is found for the MIP problem. Therefore, the exact Pythagorean fuzzy solution is $x_1 = \{(-1, 0, 1); (-2, 0, 3)\}$, $x_2 = \{(1, 2, 3); (0, 2, 4)\}$, and $x_3 = \{(-3, -2, -1); (-4, -2, 0)\}$, and the distance measure is $\mathcal{D}(\mathbb{A}X, \mathbb{B}) = 0$.

Example 5 Consider an FPFLES having the same parameters in LHS as given in Example 4 while changing the RHS values as follows:

$$\begin{aligned}
&\{(2,3,5);(1,3,6)\}x_1 \oplus \{(1,2,2);(0,2,3)\}x_2\\
&\quad\oplus\{(-3,0,1);(-5,0,2)\}x_3 = \{(-10,5,25);(-25,5,55)\}\\
&\{(-3,2,5);(-4,2,6)\}x_1 \oplus \{(4,5,6);(3,5,7)\}x_2
\end{aligned}$$

$$\oplus\{(0,2,4);(-3,2,5)\}x_3=\{(-15,8,25);(-35,8,60)\}$$
$$\{(-3,-2,-1);(-5,-2,-1)\}x_1\oplus\{(0,2,5);(-2,2,6)\}x_2$$
$$\oplus\{(-5,0,5);(-6,0,6)\}x_3=\{(-20,3,35);(-50,3,60)\}$$
$$\{(-1,0,1);(-2,0,2)\}x_1\oplus\{(0,2,4);(-2,2,4)\}x_2$$
$$\oplus\{(4,5,6);(3,5,8)\}x_3=\{(-20,-5,10);(-50,-5,25)\}$$

where x_1, x_2 and x_3 are triangular PFNs.

Since the FPFLES has a non-square structure, an MIP problem has $4 \times 3 \times 16$ inequalities found from the fuzzy multiplication, and also other constraints defined in Step 3 and Step 4. When the MIP problem is solved, there is no feasible solution, which means the FPFLES has no exact Pythagorean fuzzy solution. Thus, to find a restricted approximate solution, constraints defined in Step 6 are included in the MIP problem, and a restricted MIP problem is obtained. The restricted approximate Pythagorean fuzzy solution is found to be $x_1 = \{(-1.6667, 0.5968, 1.5741); (-2.6056, 0.5968, 3.1221)\}$, $x_2 = \{(1.3889, 2.0968, 2.8549); (1.3498, 2.0968, 3.1455)\}$, and $x_3 = \{(-3.0556, -1.8387, -0.7716); (-4.6831, -1.8387, 0.7717)\}$, and the closeness is $\mathcal{D}(\mathbb{A}X, \mathbb{B}) = 82.3844$. The relaxed approximate Pythagorean fuzzy solution can also be found by cleaning the constraints in Step 6(iii) and it is $x_1 = \{(-1.6667, 0.5968, 1.5741); (-2.6801, 0.5968, 2.9731)\}$, $x_2 = \{(1.3889, 2.0968, 2.9444); (1.1263, 2.0968, 4.1886)\}$, and $x_3 = \{(-3.0556, -1.8387, -0.8611); (-4.4596, -1.8387, 0.2875)\}$, and the closeness is $\mathcal{D}(\mathbb{A}X, \mathbb{B}) = 81.0418$.

Table 4 presents the types of solution, which vary based on the constraint arrangements, and the corresponding LHS values and distance measure between LHS and RHS for each equation.

Example 6 Consider an FPFLES having the same parameters in LHS as given in Example 4 while changing the RHS values as follows:

$$\{(2,3,5);(1,3,6)\}x_1\oplus\{(1,2,2);(0,2,3)\}x_2$$
$$\oplus\{(-3,0,1);(-5,0,2)\}x_3=\{(-5,5,15);(-10,5,20)\}$$
$$\{(-3,2,5);(-4,2,6)\}x_1\oplus\{(4,5,6);(3,5,7)\}x_2$$
$$\oplus\{(0,2,4);(-3,2,5)\}x_3=\{(-10,8,20);(-30,8,55)\}$$
$$\{(-3,-2,-1);(-5,-2,-1)\}x_1\oplus\{(0,2,5);(-2,2,6)\}x_2$$
$$\oplus\{(-5,0,5);(-6,0,6)\}x_3=\{(-15,3,30);(-45,3,55)\}$$
$$\{(-1,0,1);(-2,0,2)\}x_1\oplus\{(0,2,4);(-2,2,4)\}x_2$$
$$\oplus\{(4,5,6);(3,5,8)\}x_3=\{(-25,-5,5);(-45,-5,20)\}$$

where x_1, x_2 and x_3 are triangular PFNs.

The MIP problem is constructed by applying the solution process from Step 1 to Step 5. When the problem is solved, it is seen that there is no feasible solution, which means the FPFLES has no exact Pythagorean fuzzy solution.

Table 4 Comparison of the LHS and RHS values of each equation based on the characteristics of the solution

	LHS	RHS	Distance
Restricted Approximate			
Eq1	{(−10,5.9839,22.7469);(−16.1111,5.9839,33.287)}	{(−10,5,25);(−25,5,55)}	25.9339
Eq2	{(−15,8,25);(−21.1111,8,38.5957)}	{(−15,8,25);(−35,8,60)}	21.4043
Eq3	{(−20,3,34.5525);(−31.9136,3,43.7963)}	{(−20,3,35);(−50,3,60)}	18.5339
Eq4	{(−20,−5,10);(−33.4877,−5,12.4383)}	{(−20,−5,10);(−50,−5,25)}	16.5123
Relaxed Approximate			
Eq1	{(−10,5.9839,22.9259);(−16.1111,5.9839,33.5556)}	{(−10,5,25);(−25,5,55)}	25.4863
Eq2	{(−15,8,25.537);(−21.1111,8,39.2222)}	{(−15,8,25);(−35,8,60)}	21.3148
Eq3	{(−20,3,35);(−32.0926,3,44.3333)}	{(−20,3,35);(−50,3,60)}	17.9074
Eq4	{(−20,−5,10);(−33.6667,−5,12.5278)}	{(−20,−5,10);(−50,−5,25)}	16.3333

Table 5 Data for Example 7

Machine type	Product 1	Product 2	Product 3	Available time
Milling	{(1,2,5);(0,2,6)}	{(3,4,4);(1,4,6)}	{(0,1,2);(0,1,5)}	{(19,68,115);(5,68,213)}
Lathe	{(2,3,5);(0,3,6)}	{(0,1,11);(0,1,15)}	{(4,5,6);(2,5,8)}	{(30,77,261);(10,77,393)}
Grinder	{(2,5,7);(1,5,10)}	{(4,6,6);(2,6,9)}	{(5,7,10);(3,7,15)}	{(61,167,253);(25,167,440)}

Thus, to find a restricted approximate solution, constraints defined in Step 6 are included in the MIP problem. The restricted MIP problem is obtained, and when it is solved, no optimal solution is found, which means there is no restricted approximate Pythagorean fuzzy solution to the FPFLES. Finally, the constraints coming from Step 6(iii) are cleaned, a relaxed MIP problem is formed, and its optimal solution is found. Therefore, the relaxed approximate Pythagorean fuzzy solution is found to be $x_1 = \{(-0.7467, 0.4, 0.4); (-0.7467, 0.4, 3.7781)\}$, $x_2 = \{(1.4933, 1.9, 2.792); (0, 1.9, 4.7747)\}$, and $x_3 = \{(-2.76, -0.2213, 0.1298); (-2.76, -0.2213, 0.1298)\}$, and the closeness is $\mathcal{D}(\mathbb{A}X, \mathbb{B}) = 76.3758$. This example shows how to evaluate the approximate solution based on whether the RHS covers the LHS.

Example 7 Assume that a company manufactures three products: Product 1, Product 2, and Product 3. The machine hours required to produce each unit of the respective product and available capacity of the machines, which may restrict output, are given in Table 5. Determine how much of each product should be manufactured to utilize the entire available time (machine hours per month).

To model the manufacturing problem, the decision variables x_1, x_2, and x_3 represents the quantity of each product produced during the month. After constructing the MIP problem, it is solved and the exact Pythagorean fuzzy solution is found to be $x_1 = \{(1, 5, 7); (0, 5, 8)\}$, $x_2 = \{(6, 12, 14); (5, 12, 15)\}$, and $x_3 = \{(7, 10, 12); (5, 10, 15)\}$.

This manufacturing problem is modified from an example in the study [15] by taking the triangular fuzzy numbers as the membership value of the triangular Pythagorean fuzzy numbers. The non-membership values of triangular Pythagorean fuzzy numbers are determined considering the membership values. It is seen that the solution found in [15] overlaps with the membership values of the exact Pythagorean fuzzy solution.

Example 8 There are four products A, B, C, and D that a company sells, and the estimated cost prices for these products are $\{(200, 210, 220); (180, 210, 240)\}$,$\{(80, 100, 100); (60, 100, 120)\}$, $\{(400, 500, 500); (350, 500, 600)\}$, and $\{(190, 200, 200); (180, 200, 220)\}$ dollars, respectively. If the company sells A for a profit or loss of $s\%$ and B for a profit or loss of $r\%$, that will make a net profit of $\{(0, 100, 1860); (-2100, 100, 3900)\}$ dollars. However, if the company sells C and D at the same profit or loss percentage as A and B, that will make a net profit of $\{(0, 1000, 4080); (-3750, 1000, 9300)\}$ dollars. Identify s and r.

The given problem can be converted to the following FPFLES:

$$\begin{aligned}&\{(200, 210, 220); (180, 210, 240)\}s \oplus \{(80, 100, 100); (60, 100, 120)\}r\\&\quad = \{(0, 100, 1860); (-2100, 100, 3900)\}\\&\{(400, 500, 500); (350, 500, 600)\}s \oplus \{(190, 200, 200); (180, 200, 220)\}r\\&\quad = \{(0, 1000, 4080); (-3750, 1000, 9300)\}\end{aligned}$$

When te MIP problem is constructed and solved, the exact Pythagorean fuzzy solution is found to be $s = \{(10, 10, 15); (5, 10, 20)\}$, and $r = \{(-20, -20, -18); (-25, -20, -15)\}$. Given that $s > 0$ and $r < 0$, it is evident that the company realized a profit from the sale of A and C while incurring a loss from the sale of B and D.

This production problem is modified from an example in the study [28] by taking the triangular fuzzy numbers as the membership value of the triangular Pythagorean fuzzy numbers. It is seen that the solution found in [28] is equivalent with the membership values of the exact Pythagorean fuzzy solution.

5 Conclusion

The PFS is an expansion of intuitionistic fuzzy sets that offers an additional way of representing incomplete knowledge. This paper constructs an FPFLES, that is an FFLES in the Pythagorean fuzzy environment using triangular PFNs. The approach is based on finding the minimum and maximum elements of a finite set by formulating an MIP problem. When considering a set of finite elements, binary variables can be introduced to determine the minimum and maximum elements. These variables help model the criteria for selecting one of these elements as the minimum or maximum. This concept is implemented into fuzzy multiplication. The LHS of each equation in the FPFLES consists of the sum of the multiplications of each parameter and variable, and these parameters are taken as arbitrary PFNs. The min and max operators in the definition of fuzzy multiplication produce nonlinearity which is handled by obtaining a set of inequalities from the fuzzy multiplication. From this transformation, an FPFLES is converted into an MIP problem, resulting in identical solutions. Accordingly, the MIP problem is examined considering its optimal solution. If an optimal solution exists for the MIP problem, it means that an exact Pythagorean fuzzy solution is determined. If there is no feasible solution to the MIP problem, some rearrangements are made in the constraints and in the objective function to find an approximate Pythagorean fuzzy solution. In this paper, the approximate solution is detailed in two concepts, such as restricted and relaxed. When the solution obtained is substituted in each equation of the FPFLES, the support of each LHS value is covered by the support of each corresponding RHS value. The solution obtained from this case is called a restricted approximate Pythagorean fuzzy solution. On the other hand, in the relaxed approximate Pythagorean fuzzy solution, the support of each LHS value obtained from the substitution can be covered by the support of each corresponding RHS value or not. Besides determining the types of a solution via finding the optimal solution to the MIP problem, the distance measure is also utilized. The distance measure determines the closeness between the left-hand and right-hand sides of the equations in the FPFLES. Therefore, if the closeness is zero, then the solution found is exact; otherwise, it is an approximate solution. This study also demonstrates that when a restricted solution is obtained, the closeness is greater than the value in the case of a relaxed solution. To the best of our knowledge, this study is the first in the literature to address solving an FPFLES and categorizing the solutions into exact, restricted, and relaxed approximate. Several examples are solved to demonstrate the applicability of the adapted approach. Most of these examples have been modified from linear systems in the literature or synthetically generated. While the first two examples provide exact solutions for square FPFLESs, the third example offers an approximate solution for square FPFLES. In the fourth and fifth examples, non-square FPFLESs are considered, and exact and approximate solutions are obtained, respectively. The last example shows that evaluating the approximate solution is important based on whether the RHS values cover or do not cover the LHS values. In the future, it is planned to find a Pythagorean fuzzy solution to a dual LES in the Pythagorean fuzzy environment. On the other side, the solution of a system of nonlinear equations consisting of Pythagorean fuzzy numbers can be future work. Solving nonlinear fuzzy equation systems involves handling equations where the variables, coefficients, or both are represented by fuzzy numbers. The techniques used to solve these systems aim to find fuzzy solutions that satisfy the equations within a tolerance of the fuzziness. Numerical approximation techniques [1, 41, 45], extension principle-based methods [2, 26], or optimization-based methods [17, 25] can be some main approaches to solving such systems. Therefore, the proposed approach can be extended as an optimization-based method and implemented to solve the system of nonlinear equations.

Funding Open access funding provided by the Scientific and Technological Research Council of Türkiye (TÜBİTAK). It is declared that no funds, grants, or other support were received during the preparation of this paper. There are no institutional open-access funds available to cover the costs of publication.

Data Availability Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study. It is confirmed that the data supporting the findings of this study are available within the cited articles.

Declarations

Conflict of interest There are no Conflict of interest to declare.

References

1. Abbasbandy, S., Asady, B.: Newton's method for solving fuzzy nonlinear equations. Appl. Math. Comput. **159**(2), 349–356 (2004)
2. Abbasi, F., Allahviranloo, T., Akram, M.: A new framework for numerical techniques for fuzzy nonlinear equations. Axioms **12**(2), 222 (2023)
3. Aguilar-Peña, C., Roldán-López de Hierro, A.-F., Roldán-López de Hierro, C., Martínez-Moreno, J.: A family of fuzzy distance measures of fuzzy numbers. Soft. Comput. **20**, 237–250 (2016)
4. Ahlatcioglu, M., Albayrak, I., Kocken, H.G., Ozkok, B.A.: A mixed integer programming approach to a square fully fuzzy linear equation. J. Intell. Fuzzy Syst. **31**(3), 2009–2015 (2016)
5. Akram, M., Habib, A.: Hybridizing simulated annealing and genetic algorithms with Pythagorean fuzzy uncertainty for traveling salesman problem optimization. J. Appl. Math. Comput. **69**(6), 4451–4497 (2023)
6. Akram, M., Habib, A., Allahviranloo, T.: A new maximal flow algorithm for solving optimization problems with linguistic capacities and flows. Inf. Sci. **612**, 201–230 (2022)
7. Akram, M., Ullah, I., Allahviranloo, T., Edalatpanah, S.: Fully Pythagorean fuzzy linear programming problems with equality constraints. Comput. Appl. Math. **40**(4), 120 (2021)
8. Allahviranloo, T., Jahantigh, M., Hajighasemi, S.: A new distance measure and ranking method for generalized trapezoidal fuzzy numbers. Math. Probl. Eng. **2013**(1), 623757 (2013)
9. Allahviranloo, T., Mikaeilvand, N.: Positive solutions of fully fuzzy linear systems. (2007)
10. Allahviranloo, T., Mikaeilvand, N.: Non-zero solutions of the fully fuzzy linear systems. Appl. Comput. Math **10**(2), 271–282 (2011)
11. Arfi, B.: Linguistic Fuzzy Logic Methods in Social Sciences, vol. 253. Springer, Berlin (2010)
12. Atanassov, K.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**, 87–96 (1986)
13. Atti, H., Ben Amma, B., Melliani, S., Chadli, S (2020).: The numerical solution of intuitionistic fuzzy differential equations by the third order Runge-Kutta Nyström. Intuitionistic and Type-2 Fuzzy Logic Enhancements in Neural and Optimization Algorithms: Theory and Applications, 862, p. 119
14. Atti, H., Ben Amma, B., Melliani, S., Oukessou, M., Chadli, L.S.: Computational methods for solving intuitionistic fuzzy linear systems. *Recent Advances in Intuitionistic Fuzzy Logic Systems and Mathematics*, 217–229, (2021)
15. Behera, D., Chakraverty, S.: New approach to solve fully fuzzy system of linear equations using single and double parametric form of fuzzy numbers. Sadhana **40**, 35–49 (2015)
16. Bhardwaj, M., Sharma, J., Hu, Y.-C., Noeiaghdam, S.: Challenges and Future Directions of Fuzzy System in Healthcare Systems: A survey in Soft Computing Techniques in Connected Healthcare Systems, pp. 241–254. CRC Press, Boca Raton (2023)
17. Charfeddine, S., Boudjemline, A., Ben Aoun, S., Jerbi, H., Kchaou, M., Alshammari, O., Elleuch, Z., Abbassi, R.: Design of a fuzzy optimization control structure for nonlinear systems: a disturbance-rejection method. Appl. Sci. **11**(6), 2612 (2021)
18. Denlinger, C.: Real Analysis (International Series in Mathematics). Jones & Bartlett Publishers Inc, Massachusetts (2011)
19. Ebadi, M., Suleiman, M., Ismail, F.B., Ahmadian, A., Shahryari, M.B., Salahshour, S.: Research article a new distance measure for trapezoidal fuzzy numbers. *Mathematical Problems in Engineering*, (2013)
20. Ezzati, R., Khezerloo, S., Yousefzadeh, A.: Solving fully fuzzy linear system of equations in general form. J. Fuzzy Set Val. Anal. **1–11**, 2012 (2012)
21. Garg, H.: Linguistic interval-valued pythagorean fuzzy sets and their application to multiple attribute group decision-making process. Cogn. Comput. **12**(6), 1313–1337 (2020)
22. Garg, H.: A new possibility degree measure for interval-valued q-rung orthopair fuzzy sets in decision-making. Int. J. Intell. Syst. **36**(1), 526–557 (2021)
23. Guha, D., Chakraborty, D.: A new approach to fuzzy distance measure and similarity measure between two generalized fuzzy numbers. Appl. Soft Comput. **10**(1), 90–99 (2010)
24. Habib, A., Akram, M., Kahraman, C.: Minimum spanning tree hierarchical clustering algorithm: a new Pythagorean fuzzy similarity measure for the analysis of functional brain networks. Expert Syst. Appl. **201**, 117016 (2022)
25. Ibraheem, K.I., Khudhur, H.M.: Optimization algorithm based on the Euler method for solving fuzzy nonlinear equations. East-Eur. J. Enterp. Technol. **1**(4), 115 (2022)
26. Jafari, R., Yu, W.: Fuzzy modeling for uncertainty nonlinear systems with fuzzy equations. Math. Probl. Eng. **1**, 2017 (2017)

27. Kocken, H.G., Ahlatcioglu, M., Albayrak, I.: Finding the fuzzy solutions of a general fully fuzzy linear equation system. J. Intell. Fuzzy Syst. **30**(2), 921–933 (2016)
28. Kumar, A., Bansal, A., Babbar, N.: Fully fuzzy linear systems of triangular fuzzy numbers (a, b, c). Int. J. Intell. Comput. Cybern. **6**(1), 21–44 (2013)
29. Kumar, A., Bansal, A.: A new approach for solving fully fuzzy linear systems. Adv. Fuzzy Syst. **2011**(1), 943161 (2011)
30. Kumar, R., Edalatpanah, S., Jha, S., Singh, R.: A Pythagorean fuzzy approach to the transportation problem. Complex Intell. Syst. **5**(2), 255–263 (2019)
31. Li, D., Zeng, W.: Distance measure of pythagorean fuzzy sets. Int. J. Intell. Syst. **33**(2), 348–361 (2018)
32. Liu, H.-K.: On the solution of fully fuzzy linear systems. Int. J. Comput. Math. Sci. **4**(1), 29–33 (2010)
33. Luqman, A., Akram, M., Alcantud, J.C.R.: Digraph and matrix approach for risk evaluations under Pythagorean fuzzy information. Expert Syst. Appl. **170**, 114518 (2021)
34. Mikaeilvand, N., Noeiaghdam, Z., Noeiaghdam, S., Nieto, J.J.: A novel technique to solve the fuzzy system of equations. Mathematics **8**(5), 850 (2020)
35. Moloudzadeh, S., Allahviranloo, T., Darabi, P.: A new method for solving an arbitrary fully fuzzy linear system. Soft. Comput. **17**(9), 1725–1731 (2013)
36. Nagoorgani, A., Kavikumar, J., Ponnalagu, K.: The knowledge of expert opinion in intuitionistic fuzzy linear programming problem. Math. Probl. Eng. **1**, 2015 (2015)
37. Nayak, S., Chakraverty, S.: A new approach to solve fuzzy system of linear equations. J Math Comput Sci **7**(3), 205–212 (2013)
38. Pedrycz, W., Gomide, F.: Fuzzy Systems Engineering: Toward Human-Centric Computing. Wiley, Hoboken (2007)
39. Pradhan, R., Pal, M.: Solvability of system of intuitionistic fuzzy linear equations. Int. J. Fuzzy Logic Syst. **4**(3), 13–24 (2014)
40. Ragin, C.C., Pennings, P.: Fuzzy sets and social research. Sociol. Methods Res. **33**(4), 423–430 (2005)
41. Ramli, A., Abdullah, M.L., Mamat, M.: Broydens method for solving fuzzy nonlinear equations. Adv. Fuzzy Syst. **1**, 2010 (2010)
42. Rani, D., Garg, H.: Complex intuitionistic fuzzy preference relations and their applications in individual and group decision-making problems. Int. J. Intell. Syst. **36**(4), 1800–1830 (2021)
43. Ross, T.J.: Fuzzy Logic with Engineering Applications. Wiley, Hoboken (2005)
44. Saw, B.C., Man, S., Bairagi, A., Hazra, S.B.: Gauss-Seidel and SOR methods for solving intuitionistic fuzzy system of linear equations. Comput. Sci. Math. Forum **7**, 47 (2023)
45. Shams, M., Kausar, N., Agarwal, P., Momani, S., Shah, M.A.: Highly efficient numerical scheme for solving fuzzy system of linear and non-linear equations with application in differential equations. Appl. Math Sci. Eng. **30**(1), 777–810 (2022)
46. Shams, M., Kausar, N., Agarwal, P., Shah, M.A.: Triangular intuitionistic fuzzy linear system of equations with applications: an analytical approach. *Applied Mathematics in Science and Engineering*, 32(1), (2024)
47. Torres, A., Nieto, J.J.: Fuzzy logic in medicine and bioinformatics. Biomed. Res. Int. **2006**(1), 091908 (2006)
48. Wan, S.-P., Jin, Z., Dong, J.-Y.: Pythagorean fuzzy mathematical programming method for multi-attribute group decision making with Pythagorean fuzzy truth degrees. Knowl. Inf. Syst. **55**, 437–466 (2018)
49. Wan, S.-P., Jin, Z., Dong, J.-Y.: A new order relation for Pythagorean fuzzy numbers and application to multi-attribute group decision making. Knowl. Inf. Syst. **62**, 751–785 (2020)
50. Yager, R.: Pythagorean membership grades in multicriteria decision making. IEEE Trans Fuzzy Syst **22**, 958–965 (2014)
51. Yager, R.R.: Pythagorean membership grades in multicriteria decision making. IEEE Trans. Fuzzy Syst. **22**(4), 958–965 (2013)
52. Zadeh, L.: Fuzzy sets. Inf. Control **8**(3), 38–353 (1965)
53. Zhang, X., Xu, Z.: Extension of topsis to multiple criteria decision making with pythagorean fuzzy sets. Int. J. Intell. Syst. **29**(12), 1061–1078 (2014)