

CLOSED QUEUEING NETWORKS WITH BLOCKING

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ABSTRACT

We study the equivalencies between closed queueing networks with respect to buffer capacities and number of customers in the network. Results that relate the throughput of closed queueing networks with blocking to the throughput of closed queueing networks with infinite buffers are presented. Special cases investigated are shown to be solvable efficiently .

Key words:throughput, blocking, closed queueing networks

1.INTRODUCTION

Closed queueing networks have proved to be useful in modelling computer systems, distributed systems, production systems and flexible manufacturing systems. Several efficient algorithms to calculate the performance measures of exponential closed queueing networks with infinite buffers (CQN-I) have been developed in the literature. However, these algorithms are not applicable when limitations are imposed on buffer capacities.

In recent years, there has been a growing interest in the development of computational methods for the analysis of queueing networks with finite buffers. This is primarily due to a growing need to model actual systems which have finite capacity resources. An important feature of systems with finite buffers is the potential for a server to become blocked when the capacity limitation of the destination queue is reached. Various blocking mechanisms have been considered in the literature so far. These blocking mechanisms arose out of various studies of real life systems. A discussion on these different blocking mechanisms can be found in Onvural and Perros [7].

CQN-I have product-form steady state queue length distributions. Efficient algorithms to calculate the performance measures of these networks have been developed based on this property. In general, closed queueing networks with finite buffers (CQN-B) could not be shown to have product form solutions. However, certain CQN-B have been reported in the literature as having product form solutions in the following cases: a) when the routing matrix is reversible; b) when the probability of blocking does not depend on the number of customers in the destination node but simply is constant; c) when the service

rate at each node is constant but there is zero probability that a queue is empty; d) when the network has exactly two nodes. (see, Akyildiz[1], Kingman [6], Kelly [5], Gordon and Newell [2], and Hordijk and Van Dijk [4]). For a survey of two-node closed queueing networks with blocking see Perros [8].

We will define the network in the next section. Results that relate the throughput of CQN-B to the throughput of CQN-I and that present the equivalencies between CQN-B with respect to buffer capacities and number of customers in the network are given in Section 3. Symmetric queues are investigated in Section 4.

2. THE NETWORK

We will consider closed queueing networks consisting of N nodes and K customers. Each node has a single server and exponentially distributed service time with rate μ_i , $i=1,...,N$. B_i is the capacity of node i including the service space in front of the server. A customer upon completion of its service at node i attempts to enter destination node j with probability p_{ij} , $i=1,...,N$; $j=1,...,N$. If node j is at that moment full, the customer is forced to wait in front of the server i until there would be a space available at node j . The server remains blocked for this period of time and it can not serve any other customer waiting in its queue. If more than one server is blocked by the same node, then they will get unblocked in a first-blocked-first-unblocked fashion. Due to the blocking mechanism described above, and due to the fact that these N nodes are arbitrarily interconnected, it is possible that deadlocks may occur. For instance, assume that node i is blocked by node j . Now it is possible that a customer in node j may, upon completion of its service, choose to go to node i . If node i is at

that time full, then deadlock will occur. In this paper, it is assumed that deadlocks are detected immediately and resolved by instantaneously exchanging blocking units. This may violate the first-blocked-first-unblocked rule described above.

3.MAIN RESULTS

Let $n = \min_{i=1, \dots, N} \{B_i\}$. Clearly the number of customers in the network, K , is such

that $1 \leq K \leq \sum_{i=1}^N B_i$. For $1 \leq K \leq n$, blocking does not occur and hence the net-

work has a product form solution. This product form solution can be obtained by treating the queueing network as if the queue at each node has an infinite capacity.

Let

$x_i = e_i / \mu_i$ be the relative utilization of node i where e_i is the mean number of visits a customer makes to i th node and is given by:

$$e_i = \sum_{j=1}^N e_j p_{ji} \text{ with } e_j = 1 \text{ for some } j \quad (1)$$

$1/\mu_i$ = mean service time at node i

$(i_1, \dots, i_j, \dots, i_N)$: state of the closed queueing network with infinite buffer capacities

where i_j is the number of customers at node j with $\sum_{j=1}^N i_j = K$

$\pi(i_1, \dots, i_N)$: steady state queue length distribution of the network with infinite buffer capacities. Then, $\pi(\cdot)$ is the solution of the global balance equations :

$$\pi(i_1, \dots, i_N) \sum_{j=1}^N \sum_{k=1}^N p_{jk} \delta_j \mu_j = \sum_{j=1}^N \sum_{k=1}^N p_{jk} \delta_j \delta_k \mu_j \pi(i_1, \dots, i_j + 1, \dots, i_k - 1, \dots, i_N) \quad (2)$$

and

$$\sum_{(i_1, \dots, i_N) \in Z} \pi(i_1, \dots, i_N) = 1 \quad (3)$$

where $\delta_j = \begin{cases} 1 & \text{if } i_j > 0 \\ 0 & \text{otherwise} \end{cases}$ and Z is the set of all feasible states.

The queue length distribution of the network defined in Section 2 with infinite buffer capacities is:

$$\pi(i_1, \dots, i_N) = G_K^{-1} \prod_{j=1}^N x_j \text{ where } G_K^{-1} \text{ is a normalizing constant, chosen so that the}$$

distribution sums to unity (see, Gordon and Newell[3]).

Blocking is introduced when $K \geq n+1$ and hence product form solutions are, in general, not available. In the following theorem, we prove that the CQN-B with $K=n+1$ customers in it has a product form solution.

THEOREM 1: Let us consider a CQN-B as described in Section 2 with buffer capacities B_i . If the number of customers in the network, K , is equal to $n+1$ where $n = \min_{i=1, \dots, N} \{B_i\}$, then the network has a product form solution.

Proof: Define:

$(i_1, i_2, \dots, i_N)$: state of the CQN-B where i_j is the number of customers at node j
with $\sum_{j=1}^N i_j = n+1, i_j \leq B_j, j=1, \dots, N$

$(0, 0, \dots, i_l=1, 0, \dots, i_j=B_j+1, 0, \dots, 0)$: state of the network where $i_j=B_j+1$ denotes that node l is blocked by node j . Note that, since $K=n+1$, there can be at most one node blocked at a time.

$p(i_1, \dots, i_N)$: steady state queue length distribution of CQN-B which is the solu-

tion to the system:

-if no node is blocked

$$p(i_1, \dots, i_N) \sum_{j=1}^N \sum_{k=1}^N p_{jk} \delta_j \mu_j = \sum_{j=1}^N \sum_{k=1}^N p_{jk} \delta_j \delta_k \mu_j p(i_1, \dots, i_j + 1, \dots, i_k - 1, \dots, i_N) \quad (4)$$

-if a node is blocked

$$p(i_1, \dots, i_l, \dots, i_N) p_{lj} \mu_l = p(i_1, \dots, i_l, \dots, i_j = B_j + 1, \dots, i_N) \mu_j \quad (5)$$

Note that all nodes other than the blocked node (node l) and blocking node (node j) are empty, node l has exactly one customer and node j is full.

and

$$\sum_{(i_1, \dots, i_N) \in Z} p(i_1, \dots, i_N) = 1. \text{ where } Z \text{ is the set of all feasible states. Let,}$$

$$p(i_1, \dots, i_N) = \begin{cases} \pi(i_1, \dots, i_N) & \text{if } i_j \leq B_j, j = 1, \dots, N \\ \frac{p_{lj} e_l}{e_j} \pi(0, \dots, i_j = B_j + 1, 0, \dots, 0) & \text{if } i_l = 1 \text{ and } i_j = B_j + 1 \end{cases}$$

where e_j is the mean number of visits a customer makes to node i. Clearly,

$$\sum_{(i_1, \dots, i_N) \in Z} p(i_1, \dots, i_N) = 1. \text{ By substituting the product form solution of } \pi(\cdot) \text{'s into}$$

(4) and (5), it can be easily verified that the balance equations are satisfied.

Intuitively, this is because the space in front of the blocked server behaves like an additional space for the blocking node.

COROLLARY 1: In CQN-B, if nodes with $B_i = K-1$ are replaced by nodes with $B_i = K$ while keeping all other parameters fixed, then the new network is equivalent to the original network in the sense that they have the same rate matrix after the states in which a node is blocked by a node with $B_i = K-1$ are combined into one state.

Proof: Without loss of generality let $B_i = K-1$ for some i in CQN-B-1 and $B_i = K$ in CQN-B-2 with all other parameters are kept same. In CQN-B-1, node i in CQN-B-1 will block some node(s) while node i in CQN-B-2 will not cause any blocking. From the above discussion, we have:

$$P^{B_i=K-1}(0, \dots, i_l, \dots, B_i + l, 0, \dots, 0) = \frac{e_l p_{li}}{e_i} P^{B_i=K}(0, \dots, B_i, \dots, 0) \quad (6)$$

and

$$P^{B_i=K-1}(.) = P^{B_i=K}(.) \quad \text{if node } i \text{ is not blocking any node} \quad (7)$$

where $P^{B_i=K}(.)$, $P^{B_i=K-1}$ are the steady state queue length distribution of CQN-B-2 and CQN-B-1 respectively.

Below, we will summarize some of the well known results about the throughput of CQN-I. Let $\beta_i(K)$ be the throughput of node i and $\beta(K)$ be the throughput of the network with K customers in it. Then:

$$\beta(K) = \frac{K}{\sum_{m=1}^{K-1} \sum_{j=1}^N \beta(K-j) \sum_{i=1}^m x_i^m} \quad (8)$$

where $x_i = e_i / \mu_i$ is the relative utilization of node i . e_i is the mean number of visits a customer makes to i th node

Furthermore,

$$\beta_i(K) = \beta(K) e_i \quad i = 1, \dots, N \quad (9)$$

and $\beta(K)$ is monotonically increasing on K and bounded from above. This upper bound is given by:

$$u = \min_{i=1, \dots, N} \{p_{ij} / \mu_i, j=1, \dots, N; j \neq i\} \quad (10)$$

For CQN-B, let $\lambda_i(K)$ be the throughput of node i and $\lambda(K)$ be the throughput

of the network with K customers in it. Furthermore,

$$\lambda_i(K) = (1 - P_i^K(0) - P_i^K(b)) \mu_i \quad (11)$$

where $P_i^K(0)$ and $P_i^K(b)$ are the probabilities that node i is empty and blocked respectively with K customers in the network. Also,

$$\lambda_i(K) = \lambda(K) e_i, \quad i=1, \dots, N \quad (12)$$

where e_i is given in (1).

Clearly, $\lambda_i(K)$ depends on the parameters of the network. Four examples of $\lambda_3(K)$ as K changes from 1 to $M = \sum_{i=1}^N B_i$ for the networks given in figures 1 and 2 are given in figures 3 to 6.

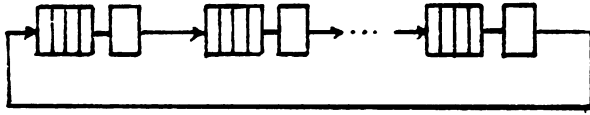


Fig. 1: Cyclic Network

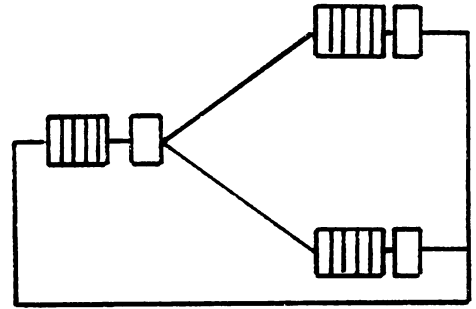
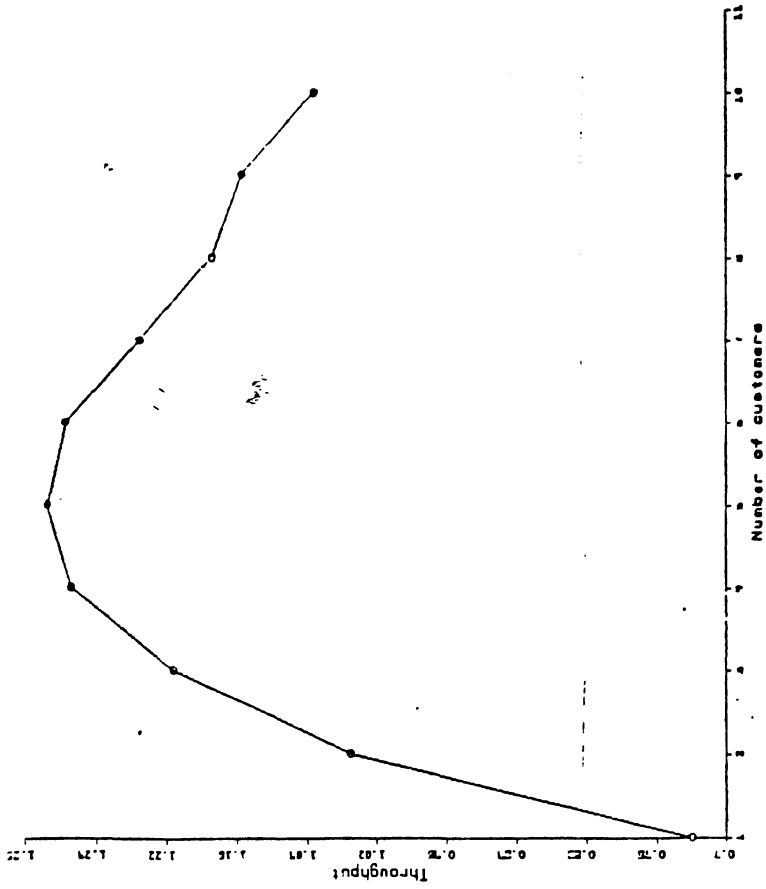
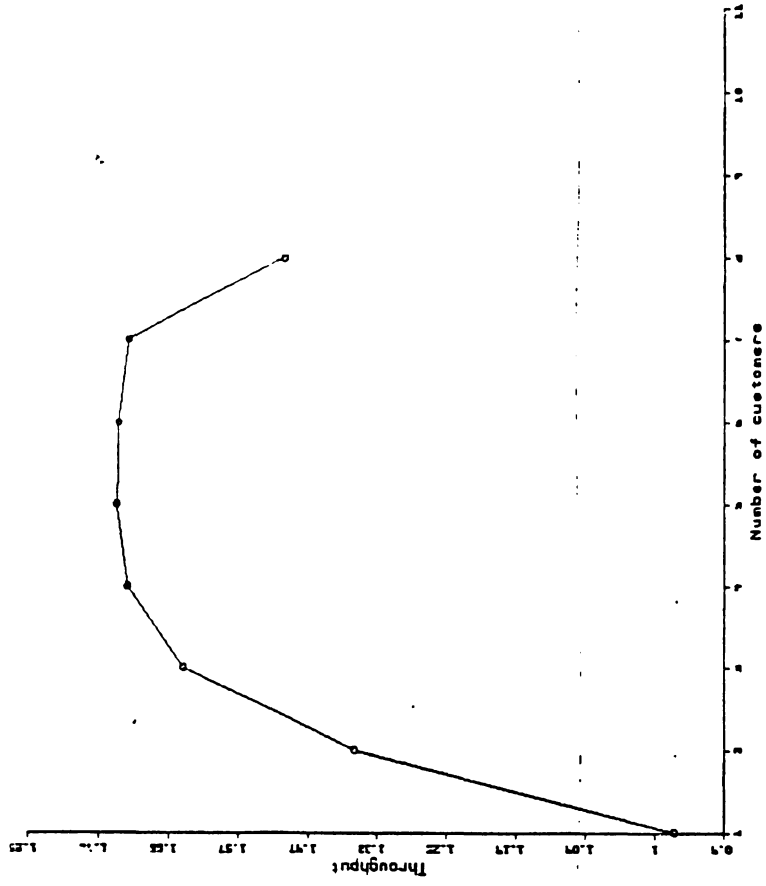
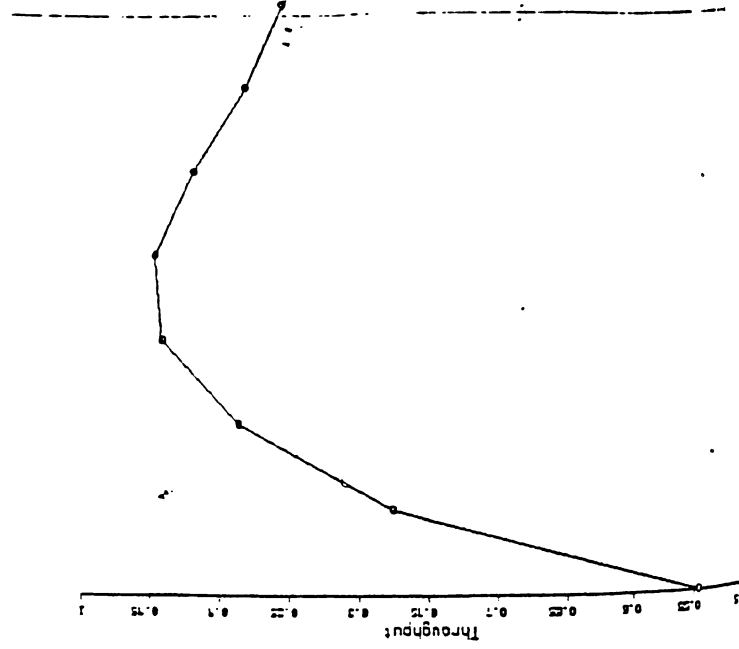
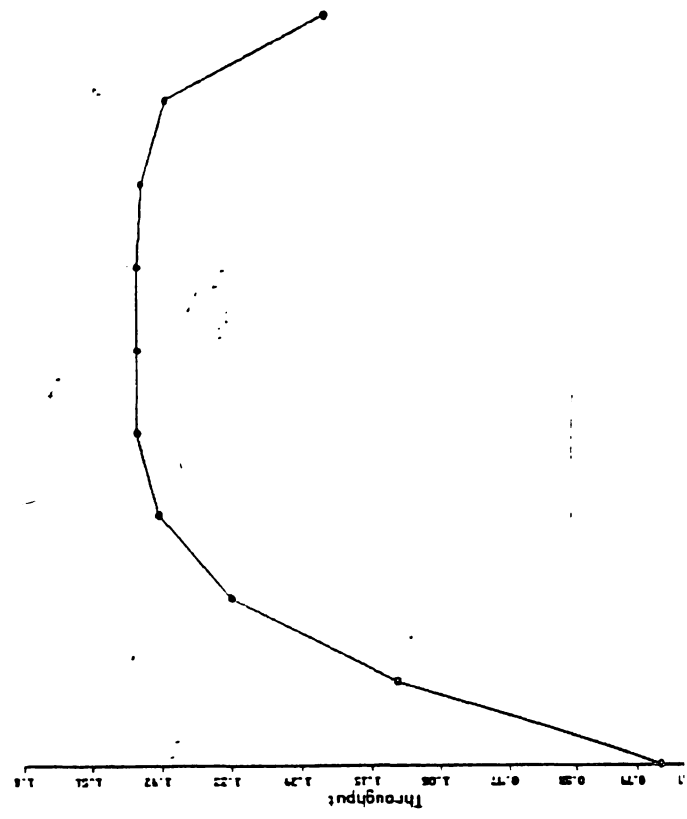


Fig. 2: Central Server Model

In these figures, $\lambda_3(K)$ increases as K increases until it reaches to some maximum value, λ^* , for some K^* , $K^* \in \{L: \lambda_3(L) \geq \lambda_3(i), i=1, \dots, M\}$ where the set can be a singleton or can have more than one element. For $K > K^*$, $\lambda_3(K)$ is non-increasing on K . This can, intuitively, be discussed as follows: $P_3^K(0)$ decreases as K increases until it reaches 0 at $K = M - B_3 + 1$. For $M - B_3 + 1 \leq K \leq M$, $P_3^K(0) = 0$. $P_3^K(b) = 0$ for $1 \leq K \leq B_1$ and non-decreasing beyond B_1 . Hence, $P_3^K(0) + P_3^K(b)$ is non-increasing in the interval $[1, B_1]$ and non-decreasing in $[M - B_3 + 1, M]$. It is not clear what happens in the interval $[B_1 + 1, M - B_3]$. Empirically, we have observed there is a point $K' \in [B_1 + 1, M - B_3]$ where $P_3^K(0) + P_3^K(b)$ continues to be non-increasing in $[B_1 + 1, K']$ and non-decreasing in $[K' + 1, M - B_3]$.



In our results, we will use the following two properties without any proof.

Property 1: $P_i^K(0)$ decreases as K increases until it reaches to zero.

Property 2: $P_i^K(b)$ increases as K increases after it becomes greater than 0.

THEOREM 2: Let $\lambda' = \max_K \{\lambda(K)\}$, $n = \min_{i=1, \dots, N} \{B_i\}$ and $M = \sum_{i=1}^N B_i$. Then

$$\beta(n+1) \leq \lambda' \leq \beta(M - \max_{i=1, \dots, N} \{B_i\} + 1)$$

Proof: For $1 \leq K \leq n+1$, the network has product form solution, hence $\beta(1) < \beta(2) < \dots < \beta(n+1)$. So, $\lambda' \geq \beta(n+1)$. To prove $\lambda' \leq \beta(M - \max_{i=1, \dots, N} \{B_i\} + 1)$,

consider node j which has the maximum buffer capacity.

For $K \geq M - \max_{i=1, \dots, N} \{B_i\} + 1$, $\lambda_i(K)$ decreases as K increases because $P_i^K(0) = 0$ and $P_i^K(b)$ increases and hence $1 - P_i^K(0) - P_i^K(b)$ decreases. Furthermore, $\lambda(K) \leq \beta(K)$, $K = 1, \dots, M$. So we have $\lambda(i) \leq \beta(M - \max_{i=1, \dots, N} \{B_i\} + 1)$,

$$i = 1, \dots, M - \max_{i=1, \dots, N} \{B_i\} + 1. \text{ Therefore, } \lambda' \leq \beta(M - \max_{i=1, \dots, N} \{B_i\} + 1).$$

COROLLARY 2: Let K' be such that $\lambda' = \lambda(K')$. Then $n+1 \leq K' \leq M - \max_{i=1, \dots, N} \{B_i\} + 1$

COROLLARY 3: Let K' be defined as in Corollary 2. Then

$$\max_{i=1, \dots, N} \left\{ \min_{j=1, \dots, N} \{B_j \text{ such that } p_{ij} \neq 0\} \right\} \leq K'.$$

Proof: Take any node i . For $K \leq K_i = \min_{j=1, \dots, N} \{B_j : \text{s.t. } p_{ij} \neq 0\}$, node i can not get blocked, therefore $P_i(b) = 0$. Therefore, throughput should increase as K increases from 1 to K_i . Furthermore, this is true for all nodes. Therefore

$$K' \geq \max_{i=1, \dots, N} K_i.$$

THEOREM 3: Let $M = \sum_{i=1}^N B_i$, $B' = \max_{i=1, \dots, N} \{B_i\}$ and $S = \{L: M - B' + 1 \leq L \leq B' + 1\}$.

If $2B' \geq M$ then the network with K customers in it has the same steady state queue length distribution for all $K \in S$.

Proof: Without loss of generality let node j be of capacity B' . Consider all nodes other than node j . Then, there is a state where all nodes are full, another state where all nodes are empty and states with all combinations in between. Hence, the transitions between these nodes are independent of K . For a state

$$(i_1, \dots, i_N), \text{ we have } i_j = K - \sum_{\substack{l=1 \\ l \neq j}}^N b_l \text{ where } b_l = \begin{cases} i_l & \text{if } i_l < B_l \\ B_l & \text{if } i_l \geq B_l \end{cases}$$

(Note that $i_l > B_l$ denotes that some node(s) are blocked by node l).

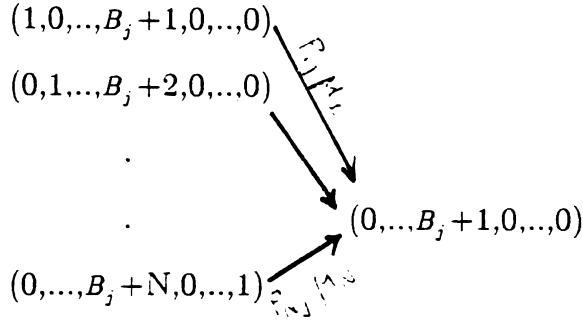
For $M - B' + 1 \leq K \leq B'$ node j can not block any node in the network and can not be empty. Hence, transitions into and out of node j is independent of K . Therefore, for $M - B' + 1 \leq K \leq B'$, we have the same rate matrix. The equivalency of states can be summarized as follows:

$P^K(i_1, \dots, i_N) = P^{K'}(i_1, \dots, i_j + K' - K, \dots, i_N)$ where $P^K(\cdot)$ and $P^{K'}(\cdot)$ are the steady state queue length distributions with K and K' customers in the network respectively.

To complete the proof, we need to show that the networks with $K = B'$ and $K = B' + 1$ customers are equivalent. However, this is immediate from Corollary 1. Hence, the networks with $K \in S$ customers has the same steady state queue length distributions.

For a better understanding of the application of Corollary 1 in the above proof

note that we only need to consider node j with $K=B_j'$ and $K=B_j'+1$ customers in the network. Let $Z_j=\{i:p_{ij} \neq 0\}$ be the set of nodes that can get blocked by node j . Furthermore, when $K=B_j'$, Z_j is empty and when $K=B_j'+1$, there can be at most one node blocked by node j at a time. Let $(0,...,i_l=1,...,i_j=B_j+1,...,0)$ denote that node l is blocked by node j , and collect all these states into one state, say $(0,...,i_j+1,...,0)$ with the rates into it as follows:



Rates out of state $(0, \dots, B_j+1, 0, \dots, 0)$ are same as rates out of state $(0, \dots, i_l, \dots, B_j+1, 0, \dots, 0)$ for $l \in Z_j$. Then, the networks with B_j' and $B_j'+1$ customers in it have the same rate matrix after all states where node j is blocking some node l , $l \in Z_j$ is combined into macro state with rates given above. Furthermore,

$$P^{B_j'+1}(0, \dots, i_l = B_j+1, 0, \dots, 0) = P^{B_j'}(0, \dots, B_j, 0, \dots, 0)$$

COROLLARY 4: If there exists a j such that $2B_j \geq M$ then increasing the buffer capacity of node j will not change the value of the maximum throughput.

THEOREM 4: Let μ_i, p_{ij} be the parameters of two closed queueing networks with buffer capacities B_i and C_i , $i=1, \dots, N$; $j=1, \dots, N$. If $\min_{i=1, \dots, N} B_i \geq l+1$ and

$$\min_{i=1, \dots, N} C_i \geq l+1, l \geq 0, \text{ then two networks with } \sum_{i=1}^N B_i - l \text{ and } \sum_{i=1}^N C_i - l \text{ customers}$$

respectively have the same rate matrix.

Proof: Note that, there is at least one customer at all nodes in both networks and the number of states are equal to each other. Let $d_i = B_i - C_i$, $i=1, \dots, N$. Then a state $(i_1, \dots, i_N)$ of the network with buffer capacities B_i is equivalent to the state $(i_1 - d_1, \dots, i_N - d_N)$ of the network with buffer capacities C_i . Equivalency here is used to state that both states have the same transition rates into and out of corresponding equivalent states.

COROLLARY 5: Let M be the total capacity of the network. Then, $\lambda(M)$ is independent of buffer capacities B_i , $i=1, \dots, N$.

COROLLARY 6: For all $K \geq \sum_{j=1}^N B_j - \max_{i=1, \dots, N} B_i + 1$, the CQN-B with buffer capacities B_i and K customers has the same rate matrix as the network with buffer capacities C_i and $K+1$ customers if $C_j = B_j + 1$ for some j and $C_i = B_i$ for $i=1, \dots, N$ and $i \neq j$.

4. SPECIAL CASES

In this section, we will study the networks with restricted parameters and give algorithms to calculate the steady state queue length distribution.

4.1. CYCLIC NETWORKS

A cyclic network is a closed queueing network consisting of tandem queues as shown in Figure 2. As defined in Section 2, the network has N nodes with following parameters:

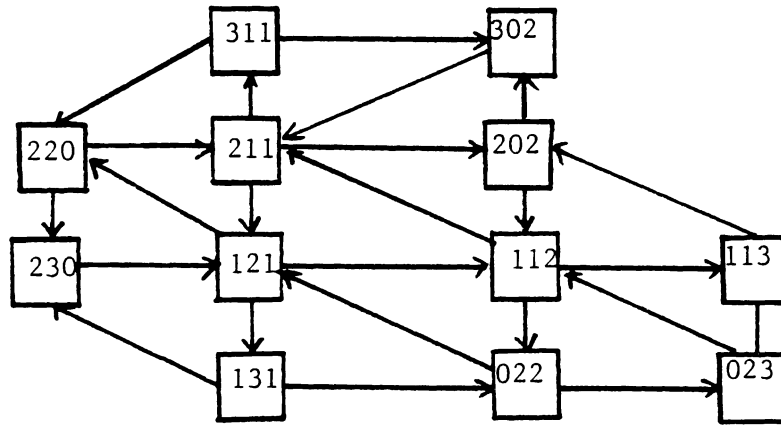
μ_i : service rate at node i . We will assume that service times are exponentially distributed

B_i : buffer capacity of node i including the space in front of the server

K: number of customers in the network

Case 1: Symmetric queues - cyclic network

Let's assume that $B_i = B < \infty$ and $\mu_i = \mu$, $i=1, \dots, N$. With these assumptions, we will show that the state space of the network with K customers in it, $1 \leq K \leq N \cdot B$, can be reduced by a factor of N. To motivate the procedure, consider a cyclic network with $B=2$, $K=4$ and $N=3$. The state space has the following structure with all transition rates equal to μ .



Solving the system numerically, we have:

$$P(2,2,0)=P(0,2,2)=P(2,0,2)=0.071429$$

$$P(2,3,0)=P(0,2,3)=P(3,0,2)=0.11905$$

$$P(2,1,1)=P(1,2,1)=P(1,1,2)=0.095238$$

$$P(3,1,1)=P(1,3,1)=P(1,1,3)=0.047619$$

This result should not be surprising if we observe that the underlying stochastic process is same for the states that have the same probabilities. i.e. nodes are indistinguishable. Hence, we can define the following classes where a state is a member of a class if that state has the same steady state probability as all the other states in the same class.

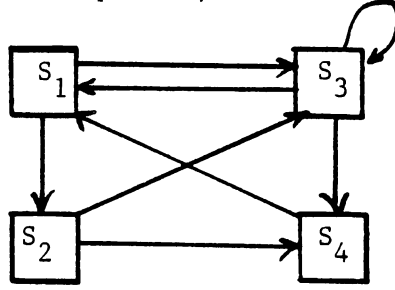
$$s_1 = \{(2,2,0), (0,2,2), (2,0,2)\}$$

$$s_2 = \{(2,3,0), (0,2,3), (3,0,2)\}$$

$$s_3 = \{(2,1,1), (1,2,1), (1,1,2)\}$$

$$s_4 = \{(3,1,1), (1,3,1), (1,1,3)\}$$

Then, we have the following state space structure for these equivalence classes with all transition rates equal to μ .



Solving this system numerically, we have:

$$P(s_1)=0.214287; P(s_2)=0.35715; P(s_3)=0.28571; P(s_4)=0.142857$$

Furthermore, $P(s_i) = \sum_{(i_1, i_2, i_3) \in S_i} P(i_1, i_2, i_3)$, $i=1, \dots, 4$. Hence, to solve the original

network, we can form the equivalence classes s_i , create the rate matrix for these classes and solve the system. Then we can obtain the queue length distribution of the original network.

The following algorithm summarizes this procedure.

ALGORITHM 1

1. Generate the equivalence classes, s_i , and set up the rate matrix.
2. Solve the system to obtain $P(s_i)$.
3. Calculate the normalizing constant, G_K , for the original network as follows:

$$G_K = \sum_{i=1}^S R_i P(s_i)$$

where S is the number of equivalence classes and R_i is the number of states in equivalence class i .

$$4.P(i_1, \dots, i_N) = G_i^{-1} P(S_i) \text{ where } (i_1, \dots, i_N) \in S_i$$

The equivalence class of a state $(i_1, \dots, i_N)$, S_i , can be found with the following algorithm:

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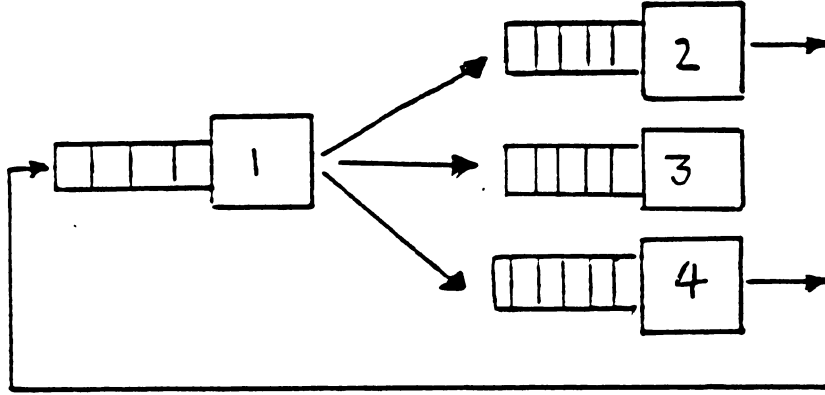
procedure equiliv(( $i_1, \dots, i_N$ ));
  for  $k := 2$  to  $N$  do
    begin
       $i := 0$ ;
      for  $l := k$  to  $N$  do
        begin
           $i := i + 1$ ;  $R(i) := i_l$ 
        end;
       $j := 1$ ;
      for  $l := i + 1$  to  $N$  do
        begin
           $R(l) := i_j$ ;  $j := j + 1$ 
        end;
       $S_i := S_i + (R(1), \dots, R(N))$ 
    end;

```

4.2. Central Server Model

A four node central server model is shown in Figure 7. Each node has a single server, buffer capacity B_i , and service times are exponentially distributed with rate μ_i . p_{1i} is the probability that a customer upon completion of its service at

node 1 attempts to go to node i , $i=2,\dots,N$.



Let's assume that $B_i = B < \infty$ and $\mu_i = \mu$, $i=2,\dots,N$. Note that μ_1 , and B_1 are not restricted. Furthermore, let $p_{1i} = 1/(N-1)$, $i=2,\dots,N$. Then nodes 2 through N are indistinguishable and hence discussions and algorithms given for case 1 can be applied to nodes 2 through N .

In Table 1, number of states in the original network and in the equivalent network is given for different values of B, K and N .

Case 3: Symmetric queues-Merge configuration

Let's assume that $B_1 < \infty$ and $B_i = \infty$, and $p_{1i} = p = 1/(N-1)$, $i=2,\dots,N$. Then, the network has the local balance property after the service rates are modified as follows:

$$\mu_i = \begin{cases} \mu_i & i=2,\dots,N \\ \mu_1 & i=1 \text{ and no node is blocked} \\ \frac{\mu_1}{p} & i=1 \text{ and some node(s) are blocked} \end{cases}$$

Hence, we have:

$$p_{\mu_1} P(i_1, \dots, i_j - 1, \dots, i_N) = \mu_j P(i_1 - 1, \dots, i_j + 1, \dots, i_N) \quad \text{if no node is blocked}$$

$$\mu_1 P(i_1 + *, \dots, i_N) = \mu_j P(i_1, \dots, i_N) \quad \text{if some node(s) are blocked.}$$

K	N	Buffer cap.	# states	# equivalence classes
4	3	(2,2,2)	12	4
5			12	4
6			7	3
5	3	(3,3,3)	18	6
6			19	7
7			18	6
7	3	(5,5,5)	33	11
8			36	12
9			37	13
10			36	12
11			33	11
12			25	9
4	4	(2,2,2,2)	31	8
5			40	10
6			40	10
7			32	8
8			15	4
7	4	(3,3,3,3)	80	
8			81	21
9			72	18
10			56	14

Table 1: Cyclic Network

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